Open Problems in Algebraic Combinatorics

(pdf compilation of blog posts)

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The rank and cranks

Submitted by ***Dennis Stanton***

The Ramanujan congruences for the integer partition function $p(n)$ (see [1]) are

$$p(5n+4) \equiv 0 \mod 5, \quad p(7n+5) \equiv 0 \mod 7, \\ p(11n+6) \equiv 0 \mod 11.$$

Dyson's rank [6] of an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots)$

$$rank(\lambda) = \lambda_1 - \lambda_1'$$

(so that the rank is the largest part minus the number of parts) proves the Ramanujan congruences

$$p(5n+4) \equiv 0 \mod 5, \quad p(7n+5) \equiv 0 \mod 7$$

by considering the rank modulo 5 and 7.

OPAC-001. *Find a 5-cycle which provides an explicit bijection for the rank classes modulo 5, and find a 7-cycle for the rank classes modulo 7.*

The generating function for the rank polynomial is known to be

$$\sum_{n=0}^{\infty} rank_n(z) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (q/z;q)_n}.$$

The rank generating function $rank_{5n+4}(z)$ for partitions of $5n+4$ does have an explicit factor of 5, but not positively. For example

$$rank_4(z) = 1 + z^{-3} + z^{-1} + z^3 + z^1 = (1 + z + z^2 + z^3 + z^4) * (1 - z + z^2)/z^3,$$

$$rank_{14}(z) = (1 + z + z^2 + z^3 + z^4) * p(z)/z^{13},$$

where $p(z)$ is an irreducible polynomial of degree 22 which has negative coeffcients. For an explicit 5-cycle which would be a rank class bijection, one would expect the

factor $1+z+z^2+z^3+z^4$ times a positive Laurent polynomial in z. Here is a conjectured modification that does this.

Definition. *For $n \geq 2$ let*

$$Mrank_n(z) = rank_n(z) + (z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}).$$

OPAC-002. *For $n \geq 0$, show that the following are non-negative Laurent polynomials in z:*

$$Mrank_{5n+4}(z)/(1+z+z^2+z^3+z^4),$$
$$Mrank_{7n+5}(z)/(1+z+z^2+z^3+z^4+z^5+z^6).$$

This conjecture says that the rank definition only needs to be changed for $\lambda = n, 1^n$ to have the "correct" symmetry. I do not know a modification which will also work modulo 11. Frank Garvan has verified OPAC-002 for $5n+4 \leq 1000$ and $7n+5 \leq 1000$.

The Andrew-Garvan [2] *crank* of a partition λ is

$$AGcrank(\lambda) = \begin{cases} \lambda_1 \text{ if } \lambda \text{ has no 1's} \\ \mu(\lambda) - (\#1\text{'s in } \lambda) \text{ if } \lambda \text{ has at least one 1,} \end{cases}$$

where $\mu(\lambda)$ is the number of parts of λ which are greater than the number of 1's of λ. For example

$$AGcrank(1111) = 0-4, \quad AGcrank(211) = 0-2, \quad AGcrank(22) = 2-0$$
$$AGcrank(31) = 1-1, \quad AGcrank(4) = 4-0.$$

The generating function of the AGcrank over all partitions of n is $AGcrank_n(z)$. For example

$$AGcrank_4(z) = z^{-4} + z^{-2} + z^2 + z^0 + z^4.$$

The generating function for the AGcrank polynomial is known to be (after modifying $AGcrank_1(z)$)

$$\sum_{n=0}^{\infty} AGcrank_n(z) q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (q/z;q)_\infty}.$$

OPAC-003. *Show*

$$AGcrank_{5n+4}(z) = (1+z^2+z^4+z^6+z^8)*(\text{a positive Laurent polynomial in } z).$$

Frank Garvan has verified OPAC-003 for $5n+4 \le 1000$.

Ramanujan factored the first 21 AGcrank polynomials, $\lambda_n = AGcrank_n(a)$, see the paper of Berndt, Chan, Chan and Liaw [5, p. 12]. Ramanujan found the factor $\rho_5 = z^4 + z^{-4} + z^2 + z^{-2} + 1$ for $n = 4, 9, 14, 19$ but the other factors did not always have positive coefficients. For example Ramanujan had

$$AGcrank_{14}(z) = (z^4+z^2+1+z^{-2}+z^{-4}) * \rho_9 * (a_5 - a_3 + a_1 + 1),$$

where

$$\begin{aligned}\rho_9 * (a_5 - a_3 + a_1 + 1) =& (z^2+z^{-2}+1)(z^3+z^{-3}+1) * (z^5+z^{-5}-z^3-z^{-3}+z+z^{-1}+1)\\ =& 3 + 1/z^{10} + 1/z^7 + 1/z^6 + 1/z^5 + 2/z^4 + 2/z^3 + 2/z^2 + 2/z\\ &+ 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}.\end{aligned}$$

A modified version of the AGcrank works for modulo 5, 7, and 11, with only the values at partitions n, 1^n changed.

Definition. *For $n \ge 2$ let*

$$MAGcrank_{n,a}(z) = AGcrank_n(z) + (z^{n-a} - z^n + z^{a-n} - z^{-n}).$$

OPAC-004. *Show that the following are non-negative Laurent polynomials in z*

$$MAGcrank_{5n+4,5}(z)/(1+z+z^2+z^3+z^4),$$
$$MAGcrank_{7n+5,7}(z)/(1+z+z^2+z^3+z^4+z^5+z^6),$$
$$MAGcrank_{11n+6,11}(z)/(1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8+z^9+z^{10}).$$

Frank Garvan has verified OPAC-004 for $tn + r \le 1000$.

The *5corecrank* (see [7]) may be defined from the integer parameters $(a_0, a_1, a_2, a_3, a_4)$ involved in the 5-core of a partition λ. Its generating function for partitions of $5n+4$ is

$$\sum_{n=0}^{\infty} q^{n+1} \sum_{\lambda \vdash 5n+4} z^{5corecrank(\lambda)} = \frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{a}\cdot\vec{1}=1\\ \vec{a}\in\mathbb{Z}^5}} q^{Q(\vec{a})} z^{\sum_{i=0}^4 i a_i}$$

where

$$Q(\vec{a}) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_i a_{i+1}, \quad a_5 = a_0.$$

Bijections for the core crank classes are known [7] for $5, 7,$ and 11.

Frank Garvan noted the following version of the previous conjectures holds for the 5corecrank for $n \leq 100,$ and $n \leq 8,$ see [3].

OPAC-005. *Show that the following are non-negative Laurent polynomials in z*

$$5corecrank_{5n+4}(z)/(1+z+z^2+z^3+z^4),$$
$$5corecrank_{5n+4,j}(z)/(1+z+z^2+z^3+z^4) \text{ when restricted to } BGcrank = j.$$

Bringmann, Ono, and Rolen [8] have proven the first statement.

References:

[1] G. Andrews, *The Theory of Partitions*. Addison-Wesley, 1976.

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[7] F. Garvan, D. Kim, and D. Stanton, Cranks and t-cores. *Invent. Math.* 101 (1990), no. 1, 1-17. **DOI: 10.1007/BF01231493**

[8] K. Ono, personal communication, July, 2019.

Sorting via chip-firing

Submitted by ***James Propp***

This post concerns various dynamical systems whose states are configurations of labeled chips on the one-dimensional integer lattice. In these configurations, multiple chips can occupy the same site on the lattice (and the "relative" position of chips at the same site is irrelevant).

The main result of [3] is the following theorem:

Theorem. *Put chips labeled* (1) *through* (n) *at site 0 on the integer lattice, and repeatedly apply moves of the form*

> *If chips* (a) *and* (b) *are both at site* k *with* $a < b$, *then slide chip* (a) *to site* $k-1$ *and chip* (b) *to site* $k+1$

until no further moves can be performed. Then if $n \equiv 0 \mod 2$, *the final configuration of the chips is independent of the moves that were made, and in particular, the chips are sorted in the sense that if* $a < b$ *then chip* (a) *is to the left of chip* (b).

Note that if $n \equiv 1 \mod 2$, there can be multiple final configurations and the chips need not end up sorted.

The following conjectures are variants of the above theorem.

OPAC-006. *Put chips labeled* (1) *through* (n) *at site 0 on the integer lattice, and repeatedly apply moves of the form*

> *If chips* (a), (b) *and* (c) *are all at site* k *with* $a < b < c$, *then slide chip* (a) *to site* $k-1$ *and chip* (c) *to site* $k+1$

until no further moves can be performed. Show that if $n \equiv 3 \mod 4$, *then the final configuration of the chips is independent of the moves that were made, and in particular, the chips are weakly sorted in the sense that if* $a < b$ *then chip* (a) *is not to the right of chip* (b).

Note: This is a special case of Conjecture 22 from [3].

OPAC-007. *Put chips labeled* (1) *through* (n) *at site 0 on the integer lattice, and repeatedly apply moves of the form*

If chips $(a), (b), (c)$ *and* (d) *are all at site* k *with* $a < b < c < d$, *then slide chips* (a) *and* (b) *to site* $k-1$ *and chips* (c) *and* (d) *to site* $k+1$

until no further moves can be performed. Show that if $n \equiv 0 \mod 4$, *then the final configuration of the chips is independent of the moves that were made, and in particular, the chips are weakly sorted.*

Note: This is a special case of Conjecture 25 from [3].

It is possible that the methods of that paper could with effort be made to solve these problems. However, I would much rather see a new and simpler approach (perhaps using the connection to root systems explored in [1] and [2]).

References:

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On the cohomology of the Grassmannian

Submitted by ***Victor Reiner***

The q-*binomial coefficient* is defined as

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q := \frac{[k+\ell]!_q}{[k]!_q[\ell]!_q}$$

where $[n]!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q$, with $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. It has many interpretations: combinatorial, algebraic, and geometric. For example, it is the *Hilbert series* for a graded ring that we will call here $R^{k,\ell}$, the cohomology ring with rational coefficients for the *Grassmannian* $\mathrm{Gr}(k, \mathbb{C}^{k+\ell})$ of k-planes in $\mathbb{C}^{k+\ell}$, with grading rescaled by half:

$$\begin{aligned} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \mathrm{Hilb}(R^{k,\ell}, q) &:= \sum_{d \geq 0} q^d \dim_{\mathbb{Q}}(R^{k,\ell})_d \\ &= \sum_{d=0}^{k\ell} q^d \dim_{\mathbb{Q}} H^{2d}(\mathrm{Gr}(k, \mathbb{C}^{k+\ell}), \mathbb{Q}). \end{aligned}$$

We know *plenty* about the structure of this ring. For example, it can be presented as the quotient of the ring of *symmetric functions* in infinitely many variables by the $\mathbb{Q}$-span of all Schur functions s_λ for which λ does not lie in a $k \times \ell$ rectangle (ℓ^k). Thus it has a $\mathbb{Q}$-basis given by $\{s_\lambda\}_{\lambda \subseteq (\ell^k)}$ and its multiplicative structure constants in $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ are the well-understood *Littlewood-Richardson coefficients*, intepreting $c^\lambda_{\mu\nu} = 0$ if $\lambda \not\subseteq (\ell^k)$. On the other hand, it also has at least two simple presentations via generators and relations:

$$\begin{aligned} R^{k,\ell} &\cong \mathbb{Q}[e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_\ell] / \left(\sum_{i+j=d} (-1)^i e_i h_j \right)_{d=1,2,\ldots,k+\ell} \\ &\cong \mathbb{Q}[e_1, e_2, \ldots, e_k]/(h_{\ell+1}, h_{\ell+2}, \ldots, h_{\ell+k}) \end{aligned}$$

where in the second line, h_r can be computed via *(dual) Jacobi-Trudi determinants*:

$$h_1 = |e_1| = e_1, \quad h_2 = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix} = e_1^2 - e_2,$$
$$h_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix} = e_1^3 - 2e_1e_2 + e_3, \text{ etc.}$$

Since $R^{k,\ell} \cong R^{\ell,k}$, we will assume from now on that $k \leq \ell$. The open problem here is to understand the Hilbert series for a tower of graded subalgebras

$$\mathbb{Q} = R^{k,\ell,0} \subset R^{k,\ell,1} \subset \cdots \subset R^{k,\ell,k} = R^{k,\ell},$$

where $R^{k,\ell,m}$ is the $\mathbb{Q}$-subalgebra of $R^{k,\ell}$ generated by all elements of degree at most m; that is, the subalgebra generated by $e_1, \ldots, e_m$. Note for $m = 0$, it is silly, as $R^{k,\ell,0} = \mathbb{Q}$, so $\mathrm{Hilb}(R^{k,\ell,0}, q) = 1$.

The $m = 1$ case is less silly. Here it turns out that

$$R^{k,\ell,1} \cong \mathbb{Q}[e_1]/(e_1^{k\ell+1}).$$

It is no surprise that $R^{k,\ell,1}$ would be a truncated polynomial algebra in the generator e_1. It was less clear why the last nonvanishing power would be $e_1^{k\ell}$, matching the top nonvanishing degree in $R^{k,\ell}$. This follows either from

- a direct calculation with the Pieri formula showing $e_1^{k\ell} = f^{(\ell^k)} s_{(\ell^k)}$ where f^λ is the (nonzero!) number of standard Young tableaux of shape λ, or
- by a special case of the Hard Lefschetz Theorem, since e_1 represents the cohomology class dual to a hyperplane section of the the smooth variety $\mathrm{Gr}(k, \mathbb{C}^{k+\ell})$ in its Plücker embedding.

As a consequence,

$$\mathrm{Hilb}(R^{k,\ell,1}, q) = 1 + q + q^2 + \cdots + q^{k\ell},$$

or equivalently, the filtration quotient $R^{k,\ell,1}/R^{k,\ell,0}$ has Hilbert series

$$\begin{aligned} \mathrm{Hilb}(R^{k,\ell,1}/R^{k,\ell,0}, q) &= \mathrm{Hilb}(R^{k,\ell,1}, q) - \mathrm{Hilb}(R^{k,\ell,0}, q) \\ &= q + q^2 + \cdots + q^{k\ell} = q \cdot [k\ell]_q = q \cdot [\ell]_q \cdot [k]_{q^\ell}. \end{aligned}$$

For small k, ℓ, m one can compute $\mathrm{Hilb}(R^{k,\ell,m}, q)$ via computer algebra, e.g., Macaulay2. One can first find a presentation for $R^{k,\ell,m}$ in terms of the generators $e_1, \ldots, e_m$, using Gröbner basis calculations that eliminate the variables

$e_{m+1}, e_{m+2}, \ldots, e_k$ from the above presentations. Given such a presentation, computer algebra lets one then compute the Hilbert series.

After doing this for small k, ℓ, m, one quickly notices that the Hilbert series for the filtration quotients $R^{k,\ell,m}/R^{k,\ell,m-1}$ are not only divisible by q^m, as forced by their definition, but also divisible by $\begin{bmatrix} \ell \\ m \end{bmatrix}_q$, which is not *a priori* obvious. Dividing these factors out leads to this conjecture.

OPAC-008. *For integers m and ℓ with $1 \leq m \leq \ell$, does the following hold?*

$$\mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q) = q^m \cdot \begin{bmatrix} \ell \\ m \end{bmatrix}_q \cdot \sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q .$$

For example, when $m = 1$, OPAC-008 predicts what we saw above:

$$\mathrm{Hilb}(R^{k,\ell,1}/R^{k,\ell,0}, q) = q^1 \cdot \begin{bmatrix} \ell \\ 1 \end{bmatrix}_q \cdot \sum_{j=0}^{k-1} q^{j\ell} \begin{bmatrix} j \\ j \end{bmatrix}_q = q \cdot [\ell]_q \cdot [k]_{q^\ell}.$$

Geanina Tudose and I were led to OPAC-008 after realizing that one of its much weaker implications [2, Conjecture 4] about $\mathrm{Hilb}(R^{k,\ell,m}, q)$ would greatly simplify the proof of the following interesting result of Hoffman, on graded endomorphisms of the cohomology ring $R^{k,\ell}$.

Theorem (Hoffmann [1]). *Let φ be a graded algebra endomorphism φ of $R^{k,\ell}$ that scales $R_1^{k,\ell}$ via some nonzero α in $\mathbb{Q}$. If $k \neq \ell$, then φ scales each component $R_d^{k,\ell}$ via α^d. If $k = \ell$, then φ has the form just described, or its composition with the involution swapping $e_r \leftrightarrow h_r$ for all r.*

This theorem was conjectured by O'Neill without the assumption that α is nonzero, motivated by a topological application: assuming it, one can easily apply the Lefschetz fixed point theorem to show $\mathrm{Gr}(k, \mathbb{C}^{k+\ell})$ has the fixed point property (i.e. every continuous self-map has a fixed point) if and only if $k \neq \ell$ and $k\ell$ is even.

In [2], one finds more background on the OPAC-008, including verification of the case $m = k$, and how the conjecture would shorten Hoffman's proof from ten pages to two pages. Here are a few more remarks.

Remark 1. Naming the inner sum in OPAC-008 as

$$f_m^{k,\ell}(q) := \sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q$$

for $0 \leq m \leq k \leq \ell$, one can check that it is defined by this recurrence

$$f_i^{k,\ell}(q) = f_{m-1}^{k-1,\ell}(q) + q^{\ell-m+1} f_m^{k-1,\ell}(q)$$

and initial conditions $f_0^{k,\ell}(q) = f_k^{k,\ell}(q) = 1$. Thus $f_m^{k,\ell}(q)$ is a q-analogue of the binomial coefficient $\binom{k}{m}$ which depends on ℓ, and has a different q-Pascal recurrence than that of $\begin{bmatrix} k \\ m \end{bmatrix}_q$.

Remark 2. One might approach OPAC-008 by finding $\mathbb{Q}$-bases of $R^{k,\ell}$ that respect the filtration by $R^{k,\ell,m}$, through Gröbner basis calculations with a lexicographic term order with $e_k > \cdots > e_2 > e_1$, and understanding the structure of the *standard monomials*. We have so far failed to make this work!

Remark 3. Recall that $R^{k,\ell}$ is the quotient of the ring of symmetric functions Λ by a certain ideal, and $R^{k,\ell,m}$ is the subalgebra of $R^{k,\ell}$ generated by its degrees up to m. The m-*Schur functions* give a $\mathbb{Q}$-basis for the subalgebra $\Lambda_{(m)}$ of Λ generated by its degrees up to m. Perhaps there is a convenient subset of m-Schur functions, for varying values of m, whose images in $R^{k,\ell}$ give a $\mathbb{Q}$-basis respecting the filtration by the $R^{k,\ell,m}$?

Remark 4. It is well-known how to generalize the cohomology ring $R^{k,\ell} = H^*(\mathrm{Gr}(k, \mathbb{C}^{k+\ell}, \mathbb{Q})$ in many directions: to other flavors of cohomology (quantum, equivariant, etc.), to other partial flag manifolds in type A, and to other Lie types. Perhaps one should approach OPAC-008 by first generalizing it in one of these directions?

References:

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The Schur cone and the cone of log concavity

Submitted by ***Dennis White***

Let C_N^k be the cone generated by products (homogeneous of degree N) of Schur functions s_λ, where $l(\lambda) \leq k$. That is, C_N^k is the set of all non-negative linear combinations of vectors of the form

$$s_{\lambda^1} s_{\lambda^2} \cdots s_{\lambda^n}$$

where $l(\lambda^i) \leq k$ and where $|\lambda^1| + \cdots + |\lambda^n| = N$

We call this cone the k-*Schur cone of degree* N. Our goal is to find the extreme vectors of this cone. That is, we wish to find products of Schur functions as above which cannot be written as a positive linear combinations of other products of that form. For instance, since $s_{3,1}s_2 = s_{3,2}s_1 + s_{1^2}s_4$, we know $s_{3,1}s_2$ is not extreme in C_6^2.

We can dispense with two easy cases immediately. When $k = 1$, since $h_i = s_i$, C_N^1 is then the cone generated by h_λ, $\lambda \vdash N$. Since the h_λ are a basis, none can be written as a linear combination of the others, so the h_λ, $\lambda \vdash N$, will be extreme.

When $k = N$, C_N^N is the cone generated by products of Schur functions. But by the Littlewood-Richardson rule, products of Schur functions are positive linear combinations of Schur functions, so s_λ, $\lambda \vdash N$ will be extreme.

When $k = 2$, the Jacobi-Trudi identity says the cone is generated by products of the form

$$h_i h_j - h_{i+1} h_{j-1} \quad \text{and} \quad h_i \quad i \geq j \geq 1.$$

We therefore call the 2-Schur cone of degree N the *cone of log-concavity*. As illustration, C_6^2 has 13 extreme vectors, which are

$$\begin{array}{ccc} s_6 & s_4 s_{1^2} & s_3 s_{2,1} \\ s_{5,1} & s_{3,1} s_{1^2} & (s_{2,1})^2 \\ s_{4,2} & s_{2^2} s_2 & s_2 (s_{1^2})^2 \\ s_{3^2} & s_{2^2} s_{1^2} & (s_{1^2})^3 \\ & s_{3,2} s_1 & \end{array}$$

Let P_k denote the partitions with $\leq k$ parts. A pair of partitions (λ, μ) in P_2 is said to be *interlaced* if it satisfies one of the following conditions:

1. $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, with $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2$;
2. $\lambda = (\lambda_1 > \lambda_2 > 0)$, $\mu = (\mu_1 > 0)$, with $\lambda_1 \geq \mu_1 \geq \lambda_2$;
3. $\lambda = (\lambda_1 > 0)$, $\mu = (\mu_1 > 0)$.

If (λ, μ) is not interlaced, it is said to be *nested*. These definitions differ somewhat from what we might usually call nested and interlaced because of the inequalities and partitions with one part.

Suppose $A = \{\alpha^1, \alpha^2, \dots\}$ is a collection of partitions in P_2, where $\sum_i |\alpha^i| = N$. We write $s_A := \prod_i s_{\alpha^i}$. These s_A are the generating vectors of the cone C_N^2.

We say A is *nested* if all pairs (α^i, α^j) in A are nested.

Theorem. *If A is not nested, then s_A is not extreme.*

Proof If A is not nested, then at least one pair (λ, μ) in A is interlaced and so satisfies one of the three interlacing conditions. Suppose that $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, with $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2$. This implies $\lambda_1 \geq \mu_1 + 1$ and $\lambda_2 - 1 \geq \mu_2$. Therefore, by Jacobi-Trudi, $s_\lambda s_\mu = s_{(\lambda_1,\mu_2)} s_{(\mu_1,\lambda_2)} + s_{(\lambda_1,\mu_1+1)} s_{(\lambda_2-1,\mu_2)}$. The other two cases follow from similar identities. See [2] for details. $\square$

Theorem. *If A is nested and all the parts of A are distinct, then s_A is extreme.*

The proof of this last theorem uses the Littlewood-Richardson rule in a non-trivial way and relies on Farkas' Lemma [1]. Farkas' Lemma states that a vector is extreme if and only if there is a hyperplane which separates it from all the other generating vectors. See [2] for details.

OPAC-009. *Show that if A is nested then s_A is extreme.*

Further information regarding this conjecture can be found in [2].

OPAC-010. *Describe the extreme vectors of C_N^3.*

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Matrix counting over finite fields

Submitted by ***Joel Brewster Lewis***

Let $r \leq m < n$ be positive integers, and q a prime power. Given a subset S of the discrete grid $\{1, \ldots, m\} \times \{1, \ldots, n\}$, one may define the *matrix count* $m_r(S; q)$ to be the number of rank-r matrices over the finite field of order q whose entries on S are equal to 0. This question concerns the properties of this matrix count as a function of q.

A first basic property is that the integer $m_r(S; q)$ is always divisible by $(q-1)^r$. (The idea of the proof is to consider the orbits formed when rescaling the rows by nonzero factors.) Consequently, it is convenient to define the *reduced* (or *projective*) *matrix count* $M_r(S; q) := \frac{m_r(S;q)}{(q-1)^r}$.

One motivation for the study of the matrix counts comes from the classical enumerative combinatorics of rook theory: the *rook number* $R_r(S)$ is the number of placements of r nonattacking rooks on $\{1, \ldots, m\} \times \{1, \ldots, n\}$ so that none of them lies on S. (Two rooks are attacking if they lie in the same row or same column, so this may equivalently be described as the number of $m \times n$ partial permutation matrices whose support is disjoint from S.) Then for any prime power q one has

$$M_r(S; q) \equiv R_r(S) \pmod{q-1},$$

(see [4, Prop 5.1]) and so one may think of $M_r(S; q)$ as a q-*analogue* of the rook number $R_r(S)$.

Depending on the diagram S, the reduced rook count may be more or less nice as a function of q. When S is a Ferrers board (i.e., the diagram of an integer partition), Haglund [2, Thm 1] showed that the function $M_r(S; q)$ is actually a polynomial in q, with positive integer coefficients, and related to the q-*rook number* of Garsia and Remmel [1]. However, when S is arbitrary, the function $M_r(S; q)$ need not be a polynomial function of q [7, Section 8.1], and in fact may be exceptionally complicated. It is natural to explore this boundary: which diagrams S give "nice" counting functions $M_r(S; q)$?

One natural way to extend Ferrers boards is to *skew shapes*, the set difference of two Ferrers boards. In fact, both are special cases of *inversion diagrams of permutations* (appearing in the literature under many names, including *Rothe diagram*): given a permutation $w = w_1 w_2 \cdots w_n$ in one-line notation, the inversion diagram contains the box (i, w_j) whenever $i < j$ and $w_j < w_i$. Then any Ferrers board S is

(for some sufficiently large n) the inversion diagram of some n-permutation, and the n-permutations whose inversion diagrams are Ferrers boards are exactly those that avoid the permutation pattern 132, i.e., those for which there do not exist $i < j < k$ with $w_i < w_k < w_j$. Similarly, every skew shape is (after rearranging rows and columns; for some sufficiently large n) the inversion diagram of a 321-avoiding permutation. In [6, Cor. 4.6], it was shown that for any permutation w with inversion diagram I_w, the matrix count $M_r(I_w; q)$ is a polynomial function of q with integer coefficients; but there exist permutations for which some of the coefficients are negative.

OPAC-011. *Prove that if w is a 321-avoiding permutation, then the matrix count $M_r(I_w; q)$ is a polynomial in q with nonnegative integer coefficients.*

This is essentially Conjecture 6.9 of [6]. It has been checked for all 321-avoiding permutations of size 14 or less.

One particular special case is worth mentioning. When n is even, the permutation $w = 214365 \cdots n(n-1)$ avoids 321; its diagram consists of exactly $n/2$ of the n diagonal boxes in $\{1, \ldots, n\} \times \{1, \ldots, n\}$. In this case, we have an explicit formula for $M_n(I_w; q)$: define the standard q-number $[n]_q := 1 + q + \ldots + q^{n-1}$ and q-factorial $[n]!_q := [1]_q \cdot [2]_q \cdots [n]_q$; then one has $M_n(I_w; q) = q^K \sum_{i=0}^{n} (-1)^i \binom{n}{i} [2n-i]!_q$ for some integer K [6, Section 6.3].

OPAC-012. *The sum $\sum_{i=0}^{n} (-1)^i \binom{n}{i} [2n-i]!_q$ is manifestly a polynomial with integer coefficients; prove that in fact the coefficients are nonnegative integers.*

OPAC-012 is essentially Conjecture 6.8 of [6]. It is easy to verify on a computer for $n \leq 40$. Ideally, one would hope for a solution method that could be applied to other cases of OPAC-011, as well.

For more open questions along these lines, see [3] and [5].

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Descents and cyclic descents

Submitted by ***Ron M. Adin*** *and* ***Yuval Roichman***

The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group $\mathfrak{S}_n$ on $[n] := \{1, 2, \ldots, n\}$ is

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

whereas its *cyclic descent set* is

$$\mathrm{cDes}(\pi) := \{1 \le i \le n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n],$$

with the convention $\pi_{n+1} := \pi_1$; see, e.g., [3, 4].

The *descent set* of a standard Young tableau (SYT) T is

$$\mathrm{Des}(T) := \{1 \le i \le n-1 \,:\, i+1 \text{ appears in a lower row of } T \text{ than } i\}.$$

For a set $I \subseteq [n-1]$ let $\mathbf{x}^I := \prod_{i \in I} x_i$ and $\mathbf{y}^I := \prod_{i \in I} y_i$. The Robinson-Schensted correspondence implies

$$(*) \qquad \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathrm{Des}(\pi)} \mathbf{y}^{\mathrm{Des}(\pi^{-1})} = \sum_{\lambda \vdash n} \sum_{P,Q \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(Q)} \mathbf{y}^{\mathrm{Des}(P)},$$

where the first summation in the RHS is over all partitions of n, and $\mathrm{SYT}(\lambda)$ denotes the set of all SYT of shape λ.

OPAC-013*. Find a cyclic analogue of Equation (*).*

As a first step, note that Equation (*) implies

$$(**) \qquad \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\lambda \vdash n} \#\mathrm{SYT}(\lambda) \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}.$$

Definition *[1]. Let $n \ge 2$, and let $\mathcal{T}$ be any finite set equipped with a* descent map $\mathrm{Des} : \mathcal{T} \to 2^{[n-1]}$. *Consider the cyclic shift* $\mathrm{sh} : [n] \to [n]$, *mapping i to* $i + 1 \pmod{n}$, *extended naturally to* $2^{[n]}$. *A* cyclic extension *of the descent map* Des *is a*

pair (cDes, p)*, where* $\mathrm{cDes} : \mathcal{T} \to 2^{[n]}$ *is a map and* $p : \mathcal{T} \to \mathcal{T}$ *is a bijection, satisfying the following axioms: for all* $T \in \mathcal{T}$*,*

- *(extension)* $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$*,*
- *(equivariance)* $\mathrm{cDes}(p(T)) = \mathrm{sh}(\mathrm{cDes}(T))$*,*
- *(non-Escher)* $\varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n]$*.*

For example, letting $\mathcal{T} = \mathfrak{S}_n$ be the symmetric group, the map $\mathrm{cDes}(\pi)$ defined above and the rotation $p([\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n]) := [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}]$ determine a cyclic extension of the map $\mathrm{Des}(\pi)$ defined above.

A cyclic extension of the tableaux descent map $\mathrm{Des}(T)$ defined above, for SYT of rectangular shapes, was introduced in [9]. In fact, this descent map on $\mathrm{SYT}(\lambda/\mu)$ has a cyclic extension if and only if the skew shape λ/μ is not a connected ribbon [1, Theorem 1.1]; a constructive proof of this result was recently given in [7]. All cyclic extensions of Des on $\mathrm{SYT}(\lambda/\mu)$ share the same distribution of cDes.

The following cyclic analogue of (**) was proved in [1, Theorem 1.2]:

$$(***) \qquad \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathrm{cDes}(\pi)} = \sum_{\substack{\text{non-hook} \\ \lambda \vdash n}} \#\mathrm{SYT}(\lambda) \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{cDes}(T)} + \sum_{k=1}^{n-1} \binom{n-2}{k-1} \sum_{T \in \mathrm{SYT}((1^k) \oplus (n-k))} \mathbf{x}^{\mathrm{cDes}(T)}$$

OPAC-014*. Find a Robinson-Schensted-style bijective proof of Equation (***).*

By a classical theorem of Gessel and Reutenauer [5, Theorem 2.1], there exists a collection of non-negative integers $\{m_{\lambda,\mu}\}_{\lambda,\mu \vdash n}$ such that for every conjugacy class C_μ of type μ in $\mathfrak{S}_n$

$$(\dagger) \qquad \sum_{\pi \in C_\mu} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\lambda \vdash n} m_{\lambda,\mu} \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}.$$

OPAC-015*. Find a bijective proof of Equation (†).*

A bijective proof of a cyclic extension of Equation (†), like the one given in [6, Theorem 6.2], is also desired.

Thrall [11] asked for a description of the coefficients of in Equation (†); for recent discussions see, e.g., [8, 2, 10]. Particularly appealing is a combinatorial interpretation of $m_{\lambda,\mu}$ as the cardinality of a nice set of objects. This has been done in some special cases – for example, when λ is a hook-shaped partition:

$$m_{(n-k,1^k),\mu} = \#\{\pi \in C_\mu \,:\, \mathrm{Des}(\pi) = [k]\} \qquad (0 \le k \le n-1).$$

OPAC-016. *For which partitions* $\mu \vdash n$ *is the sequence* $\{m_{(n-k,1^k),\mu}\}_{k=0}^{n-1}$ *unimodal?*

It is known that this sequence is unimodal for $\mu = (n)$, and conjecturally the same holds for all rectangular shapes $\mu = (r^s)$; see [6].

Unlike the full symmetric group, when restricted to a general conjugacy class the definition of $\mathrm{cDes}(\pi)$ given above does not yield a cyclic extension of Des. However, the following holds.

Theorem *[6, Theorem 1.4]. The descent map* Des *on a conjugacy class* C_μ *of* $\mathfrak{S}_n$ *has a cyclic extension* (cDes, p) *if and only if the partition* μ *is* not *of of the form* (r^s) *for a square-free* r.

The proof involves higher Lie characters and does not provide an explicit description of the extension.

OPAC-017. *Find an explicit combinatorial description for the cyclic extension of* Des *on a conjugacy class of* $\mathfrak{S}_n$, *whenever such an extension exists.*

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Root polytope projections

Submitted by ***Sam Hopkins***

Let Φ be a crystallographic root system in an Euclidean vector space V with inner product $\langle \cdot, \cdot \rangle$. For any subspace $U \subseteq V$ let $\pi_U \colon U \to V$ denote the orthogonal (with respect to $\langle \cdot, \cdot \rangle$) projection. We call a nonzero subspace $\{0\} \neq U \subseteq V$ a Φ-*subspace* if $\Phi \cap U$ spans U. In this case $\Phi \cap U$ is a (crystallographic) root system in U.

Recall that the poltyope $\mathrm{ConvHull}(\Phi)$, which is the convex hull of all the roots, is called the *root polytope* of Φ (see, e.g., [1]). Let $\{0\} \neq U \subseteq V$ be a Φ-subspace. Define $\kappa(\Phi, U)$ to be the minimal $\kappa \geq 1$ for which

$$\pi_U(\mathrm{ConvHull}(\Phi)) \subseteq \kappa \cdot \mathrm{ConvHull}(\Phi \cap U).$$

In other words, $\kappa(\Phi, U)$ is how much we need to dilate the root polytope of $\Phi \cap U$ by to contain the projection of the root polytope of Φ.

Example. Let V be $\mathbb{R}^4$ with its standard orthonormal basis e_1, e_2, e_3, e_4. Let

$$\Phi = \{\pm(e_i - e_j), \pm(e_i + e_j) \colon 1 \leq i < j \leq 4\},$$

i.e., $\Phi = D_4$. Let us use the notation

$$\alpha_1 := e_1 - e_2, \quad \alpha_2 := e_2 - e_3, \quad \alpha_3 := e_3 - e_4, \quad \alpha_4 := e_3 + e_4.$$

Let $U := \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \alpha_3, \alpha_4\}$. Note that $U \subseteq V$ is the subspace orthogonal to $\omega_2 := e_1 + e_2$. Thus for instance we can compute

$$\pi_U(\alpha_2) = \alpha_2 - \frac{\langle \alpha_2, \omega_2 \rangle}{\langle \omega_2, \omega_2 \rangle}\omega_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_2 - e_3 = -\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_3 - \frac{1}{2}\alpha_4.$$

In fact, the projection

$$\pi_U(\Phi) = \{\pm\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_3 \pm \tfrac{1}{2}\alpha_4, \pm\alpha_1, \pm\alpha_3, \pm\alpha_4\}$$

consist of 14 points. On the other hand, it is easy to see that

$$\Phi \cap U = \{\pm\alpha_1, \pm\alpha_3, \pm\alpha_4\}.$$

Thus $\pi_U(\mathrm{ConvHull}(\Phi))$ is a rhombic dodecahedron, and $\mathrm{ConvHull}(\Phi \cap U)$ is an octahedron inscribed inside this rhombic dodecahedron. So $\kappa(\Phi, U) = \frac{3}{2}$.

In [2, Lemma 2.10] it is shown that $\kappa(\Phi, U) < 2$ (and this exact value 2 turns out to be important for applications in that paper). However, the proof there unfortunately ultimately relies on a case-by-case analysis, leading to the following open problem:

OPAC-018. *Prove in a uniform way (i.e., without relying on the classification of root systems) that* $\kappa(\Phi, U) < 2$.

Let $\kappa(\Phi)$ be the max of $\kappa(\Phi, U)$ over all Φ-subspaces U. It turns out that $\kappa(\Phi)$ can get arbitrarily close to 2. Indeed $(2 - \kappa(\Phi))$ is on the order of $\mathrm{rank}(\Phi)^{-1}$ (see [2, Table 8]). The irreducible root system which minimizes $(2 - \kappa(\Phi)) \times \mathrm{rank}(\Phi)$ is $\Phi = E_8$ for which this quantity is equal to $\frac{8}{30}$.

OPAC-019. *Give a root system-theoretic interpretation of* $\kappa(\Phi, U)$ *or* $\kappa(\Phi)$ *(e.g., in terms of other fundamental invariants like the rank of the root system, the Coxeter number, the degrees, the index of connection, et cetera).*

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The restriction problem

Submitted by ***Mike Zabrocki***

A representation of $Gl_n(\mathbb{C})$ is a homomorphism ϕ from $Gl_n(\mathbb{C})$ to $Gl_k(\mathbb{C})$. The value of k is the dimension of the representation.

Up to isomorphism, there is one irreducible polynomial $Gl_n(\mathbb{C})$ representation for each partition λ with the length of λ less or equal to n. The character of that irreducible representation is the Schur function $s_\lambda(x_1, x_2, \ldots, x_n)$ indexed by the partition λ. The dimension of that representation is the number of column strict tableaux of shape λ with entries in $\{1, 2, \ldots, n\}$.

Since the permutation matrices are a natural subgroup of $Gl_n(\mathbb{C})$, when an irreducible $Gl_n(\mathbb{C})$ representation is restricted from $Gl_n(\mathbb{C})$ to S_n it decomposes as a direct sum of irreducible representations.

The *restriction problem* is the following:

OPAC-020. *Find a combinatorial description of the decomposition of the irreducible $Gl_n(\mathbb{C})$ module indexed by the partition λ into symmetric group S_n-irreducibles.*

This problem has a very long history, but generally very few people publish partial progress or failed attempts so there is very little written about it after the 1980's beyond special cases.

To determine how a $Gl_n(\mathbb{C})$ irreducible decomposes into S_n irreducibles we can use the character $s_\lambda(x_1, x_2, \ldots, x_n)$ of the irreducible $Gl_n(\mathbb{C})$ module and its evaluation at eigenvalues of permutation matrices $S_n \subseteq Gl_n(\mathbb{C})$. Let γ be a partition of n and $(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n})$ be the eigenvalues of a permutation matrix of cycle structure γ (up to reordering, this list only depends on the cycle structure).

If we evaluate the symmetric function $s_\lambda(x_1, x_2, \ldots, x_n)$ at the eigenvalues $(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n})$, this is the value of the S_n character at a permutation of cycle structure γ.

Representation theory provides a formula for the multiplicity for a symmetric group irreducible indexed by μ (where μ is a partition of n and the character of this irreducible is denoted χ^μ). It is equal to

$$A_{\lambda,\mu} := \sum_\gamma s_\lambda(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n})\chi^\mu(\gamma)/z_\gamma$$

where the sum is over all partitions γ of n.

Computing a few examples of this formula should indicate why it is not a particularly satisfactory answer beyond as a means of arriving at a numerical value. Littlewood [2, 9] showed in the 50's that the multiplicity can be computed using the operation of plethysm:

$$A_{\lambda,\mu} = \langle s_\lambda, s_\mu[1 + s_1 + s_2 + \cdots]\rangle$$

This is an advance in the problem, but recasts the solution of one problem in terms of another for which we don't have a combinatorial formula.

I first became interested in this problem in the early 2000's because, from time to time, I would encounter a module for which a formula for the $Gl_n(\mathbb{C})$ character was well known, but the symmetric group module structure was not. Then in 2016, Rosa Orellana and I [5] found a basis of the symmetric functions that are the the characters of the symmetric group as permutation matrices $S_n \subseteq Gl_n(\mathbb{C})$ in the same way that the Schur functions are characters of $Gl_n(\mathbb{C})$. That, is there is a basis $\{\tilde{s}_\mu\}$ (and one could take the following formula as a definition of this basis) such that for all n sufficiently large,

$$s_\lambda = \sum_{\mu:\ |\mu|\leq|\lambda|} A_{\lambda,(n-|\mu|,\mu)}\tilde{s}_\mu.$$

Then, for γ a partition of n, we have $\tilde{s}_\mu(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n}) = \chi^{(n-|\mu|,\mu)}(\gamma)$.

For each partition λ, following symmetric function encodes all of the values of the symmetric group character of this representation:

$$F_{\lambda,n} := \sum_\gamma s_\lambda(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n})\frac{p_\gamma}{z_\gamma}$$

where the sum is over all γ partitions of n. An answer to the restriction problem would provide a Schur expansion of this expression as a symmetric function of degree n. Note that if $\ell(\lambda) > n$, then $F_{\lambda,n} = 0$.

Programs for computing data are easily accessible in Sage [7, 8] through the ring of symmetric functions. For instance, the following code:

```
sage: s = SymmetricFunctions(QQ).schur()
sage: s[3].character_to_frobenius_image(4)
```

```
s[2, 1, 1] + s[2, 2] + 4*s[3, 1] + 3*s[4]
```
computes the Schur expansion of $F_{(3),4}$ by evaluating the character $s_{(3)}(x_1, x_2, x_3, x_4)$ at the eigenvalues of permutation matrices and computing the Schur expansion of that expression.

In the case when $\lambda = (r)$, we have the following, which should be a special case of what the answer might look like in general:

Proposition. *(Reformulation of [1]; see Exercise 7.73 of [10]; MacMahon's Master Theorem [4] can be used to derive this.) The coefficient of the Schur function s_μ in $F_{(r),n}$ (where μ is a partition of n) is equal to the coefficient of q^r in the Schur function evaluation $s_\mu(1, q, q^2, \ldots)$.*

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A localized version of Greene's theorem

Submitted by ***Joel Brewster Lewis***

Here is one collection of permutation statistics associated to a permutation w in the symmetric group $\mathfrak{S}_n$, viewed as a sequence containing each element of $\{1, \ldots, n\}$ exactly once: for any $k \geq 0$, let A_k be the maximum size of the disjoint union of k increasing subsequences of w. For example, if $w = 236145$, then $A_0 = 0$, $A_1 = 4$ (witnessed uniquely by the subsequences 2345), $A_2 = 6$ (witnessed uniquely by the pair of subsequences $\{236, 145\}$), and $A_k = 6$ for all $k > 2$. Similarly, one can define a second collection D_k of permutation statistics by instead taking decreasing subsequences; with $w = 236145$ one has $D_0 = 0$, $D_1 = 2$, $D_2 = 4$, $D_3 = 5$, and $D_k = 6$ for all $k > 3$. The following paraphrase of a famous theorem of Greene explains how these sequences are related to each other.

Theorem *(Greene [1, Thm. 3.1]). Let w be a permutation in $\mathfrak{S}_n$, with A_k, D_k as above. For $k \geq 1$, let $\lambda_k := A_k - A_{k-1}$ and $\mu_k := D_k - D_{k-1}$. Then the sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ are weakly decreasing sequences of nonnegative integers with sum n (that is, they are integer partitions of n); in fact, they are conjugate partitions, in the sense that μ_i is equal to the number of parts of λ of size larger than or equal to i, and vice-versa.*

Of course the excitement of the theorem is not just that λ and μ are any pair of conjugate partitions, but that they are a particularly meaningful pair: λ is exactly the shape of the standard Young tableau associated to w by the Robinson-Schensted correspondence.

We now describe a "localized" version of the quantities A_k and D_k.

An *ascent* in a sequence $u = (u_1, u_2, \ldots)$ is an index i such that $u_i < u_{i+1}$. Let $\mathrm{asc}(u)$ denote the number of ascents of u, and let $\mathrm{asc}^*(u) := \begin{cases} 0 & \text{if u is empty,} \\ 1 + \mathrm{asc}(u) & \text{otherwie.} \end{cases}$.

Given a permutation w in the symmetric group $\mathfrak{S}_n$, define

$$A'_k := \max_{u_1, \ldots, u_k} \left(\mathrm{asc}^*(u_1) + \cdots + \mathrm{asc}^*(u_k)\right)$$

where the maximum is taken over disjoint subsequences u_i of w. For example, with $w = 63417285$, one has $A'_0 = 0$, $A'_1 = \mathrm{asc}^*(w) = 4$, $A'_2 = 7$ (one can take subsequences 6125 and 3478), and $A'_k = 8$ for all $k \geq 3$ (one can take subsequences 67, 348, 125, among many other options). On the other hand, for a sequence u, define $d(u)$ to be the longest decreasing subsequence of u, and define

$$D'_k = \max_{w=u_1|\cdots|u_k} (d(u_1) + \cdots + d(u_k))$$

where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of subsequences (now obliged to be consecutive). For example, with $w = 63417285$, one has $D'_0 = 0$, $D'_1 = d(w) = 3$, $D'_2 = 5$ (witnessed by $6341 \mid 7285$, among other divisions), $D'_3 = 7$ (witnessed by $6341 \mid 72 \mid 85$) and $D'_k = 8$ for $k \geq 4$ (witnessed by $63 \mid 41 \mid 72 \mid 85$).

The following theorem shows that these localized versions are again closely related.

Theorem *(Lewis–Lyu–Pylyavskyy–Sen [3, Lem. 2.1]). Let w be a permutation in $\mathfrak{S}_n$, with A'_k, D'_k as above. For $k \geq 1$, let $\lambda'_k := A'_k - A'_{k-1}$ and $\mu'_k := D'_k - D'_{k-1}$. Then the sequences $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ and $\mu' = (\mu'_1, \mu'_2, \ldots)$ are weakly decreasing sequences of nonnegative integers with sum n (that is, they are integer partitions of n); in fact, they are conjugate partitions, in the sense that μ'_i is equal to the number of parts of λ' of size larger than or equal to i, and vice-versa.*

Again, the excitement of the theorem has something to do with the specific meaning of the partition. In this case, μ' is the *soliton partition* describing the long-term behavior of a *multicolor box-ball system (BBS)* initialized with one ball in each color $\{1, \ldots, n\}$, arranged according to w. Here the BBS is a dynamical system consisting of balls in an infinite strip; balls take turns jumping to the first available cell, beginning with the largest-numbered ball. For example, using 0's to denote empty cells and beginning with the initial configuration $63417285000\cdots$, one BBS move (in which all balls jump once, starting with ball 8 and ending with ball 1) results in the new position $0030617485200\cdots$. A second move produces the configuration $0003006174085200\cdots$, and a third move produces the configuration $0000300061740085200\cdots$. At every subsequent time-step, the three balls 852 advance three steps to the right, the two pairs 74 and 61 advance two steps to the right, and the singleton 3 advances one step to the right. These unchanging sequences are the solitons, and the soliton partition $(3, 2, 2, 1)$ records their length. Not only does this partition equal μ, but the proof of the above theorem uses the box-ball dynamics in an essential way, by suitably interpreting the statistics A'_k and D'_k for more general BBS configurations, showing that they are preserved under a step of the system, and showing that they give conjugate partitions once the system has decomposed into solitons.

OPAC-021. *Give a direct proof of the above theorem concerning A'_k and D'_k that stays inside the realm of permutation combinatorics (i.e., not using the full machinery of the box-ball system).*

(It is not difficult but also not trivial to prove that λ'_1 is the number of positive parts of μ' and vice-versa. It is also not too hard to show an inequality between λ' and the conjugate of μ in dominance order.)

Greene's invariants A_k, D_k may be defined more generally for any finite poset P, by considering maximum collections of chains and antichains [2]. These are called the *Greene-Kleitman invariants* of the poset. (One recovers the permutation case by considering a

permutation as a certain poset of dimension 2.) It is natural to ask the same for the localized versions.

OPAC-022. *Is there a "localized" version of the Greene-Kleitman invariants that specializes to the quantities* A'_k, D'_k *in some case naturally associated to permutations?*

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Descent sets for tensor powers

Submitted by ***Bruce W. Westbury***

Let T be a standard tableau of size n. The *descent set* of T, $\mathrm{Des}(T)$, is the subset of $[n-1] := \{1, 2, \ldots, n-1\}$ consisting of those i for which $i+1$ appears in a lower row than i.

For each subset $D \subseteq [n-1]$, we have a fundamental quasisymmetric function F_D. (See [5, Ch. 7] for background on symmetric and quasisymmetric functions.) A basic fact is that the combinatorics of descents gives the quasisymmetric expansions of the Schur functions. Let s_λ be the Schur function associated to the partition λ. Then, for all partitions λ, we have the expansion

$$s_\lambda = \sum_{T\,:\,\mathrm{sh}(T)=\lambda} F_{\mathrm{Des}(T)}.$$

Let V be a highest weight representation of a reductive algebraic group or Lie algebra. For each highest weight λ we have an irreducible representation $V(\lambda)$. Then, for each $n \geq 0$, we have the decomposition

$$\otimes^n V = \sum_\lambda U(n,\lambda) \otimes V(\lambda)$$

where $U(n,\lambda)$ is the space of highest weight tensors of weight λ.

Each isotypic subspace, $U(n,\lambda)$, has a natural action of the symmetric group $\mathfrak{S}_n$ and hence a Frobenius character, $\mathrm{ch}(n,\lambda)$. The problem is to find the quasisymmetric expansion of this symmetric function.

Let C be the crystal of V. Then, for each $n \geq 0$, we have the decomposition

$$\otimes^n C = \sum_\lambda W(n,\lambda) \times C(\lambda)$$

where $W(n,\lambda)$ is the set of highest weight words of weight λ.

A *descent set* is a function $\mathrm{Des} : W(n,\lambda) \to 2^{[n-1]}$ such that

$$\mathrm{ch}(n,\lambda) = \sum_{T \in W(n,\lambda)} F_{\mathrm{Des}(T)}$$

It is clear that descent sets in this sense exist since the quasisymmetric expansion of $\mathrm{ch}(n,\lambda)$ corresponds to a multiset of subsets of $[n-1]$ whose cardinality is the cardinality of $W(n,\lambda)$. However the problem is to give a construction.

OPAC-023. *Give an explicit construction of descent sets for various representations V.*

Here is a classical example: take V to be the vector representation of $GL(V)$. Then, by Schur-Weyl duality, we can identify $W(n,\lambda)$ with the set of standard tableaux of shape λ, and the aforementioned combinatorial definition of the descent set of a standard tableau gives us a descent set in this sense.

The current situation is that descent sets are only known for the vector representations of classical groups; that is, for the vector representation of a general linear group (as just explained), for the vector representation of a symplectic group [1], and for the vector representation of an orthogonal group [2, 3].

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From Schensted to Pólya

Submitted by ***Dennis White***

Suppose G is a finite permutation group acting on $[n] := \{1, 2, \ldots, n\}$. Let C_ρ denote the conjugacy class of permutations of type ρ in symmetric group S_n. Let $\chi^\lambda(\rho)$ be the λ irreducible S_n character evaluated at the conjugacy class C_ρ.

Define

$$K_{\lambda,G} := \sum_\rho \frac{|G \cap C_\rho|}{|G|} \chi^\lambda(\rho)$$

In fact, $K_{\lambda,G}$ is the number of occurrences of the irreducible λ in the induction of the trivial character of G up to S_n, or, by Frobenius reciprocity, the dimension of the G-fixed space inside the S_n-irreducible corresponding to λ.

It is therefore an integer and

$$K_{\lambda,G} \leq f^\lambda$$

where f^λ is the number of standard Young tableaux (SYT) of shape λ.

OPAC-024. *Interpret $K_{\lambda,G}$ as a subset of SYT of shape λ.*

The group G then acts (Pólya action) on colorings of $[n]$. For a partition μ, let Δ_μ be the orbits of μ-colorings of $[n]$, that is, the orbits in which color i appears μ_i times. It follows from Pólya's Theorem that

$$(*) \qquad |\Delta_\mu| = \sum_\lambda K_{\lambda,\mu} K_{\lambda,G}$$

where $K_{\lambda,\mu}$, the Kostka number, counts the number of semistandard Young tableaux (SSYT) of type μ and shape λ.

Alternatively (see [1]), compute the dimension of the μ-weight space inside the G-fixed space of the $GL(V)$-representation on $V^{\otimes n}$ in two ways, either directly or via Schur-Weyl duality.

OPAC-025. *Give a Schensted-like proof of Equation (*).*

Example 1. If $G = S_\nu = S_{\nu_1} \times S_{\nu_2} \times \cdots$ (a Young subgroup), then $K_{\lambda,G} = K_{\lambda,\nu}$ and

$$|\Delta_\mu| = \sum_\lambda K_{\lambda,\mu} K_{\lambda,\nu}.$$

The tableaux in OPAC-024 are then the standarization tableaux of the SSYT and the Schensted-like proof is the Robinson-Schensted-Knuth correspondence (RSK).

Example 2. We say i is a *descent* in SYT T if i lies in a row above $i+1$ in T. Define

$$\mathrm{maj}(T) := \sum_{i \text{ descent of } T} i.$$

If $G = Z_n$, the cyclic group of order n, acting on $[n]$, then the tableaux of OPAC-024 are those SYT T such that $\mathrm{maj}(T)$ is a multiple of n. See [1]. However, usual Schensted applied to these tableaux does not produce orbit representatives for the Pólya action, so OPAC-025 is unresolved.

Example 3. The techniques in [1] can also be used to solve OPAC-024 if $G = Z_{n-1}$ acting on $[n]$.

Example 4. Also solved in [1] is the case where G is the alternating subgroup of a Young subgroup. If G is the alternating subgroup of S_ν, then $K_{\lambda,G} = K_{\lambda,\nu} + K_{\lambda',\nu}$. Here, λ' denotes the conjugate of λ. The solution to OPAC-025 uses a small modification to the standardization argument for the RSK algorithm for the Young subgroup case.

Example 5. In fact, suppose $G \times H$ acts on $[a+b]$, with G acting on $[a]$ and H on $[b]$ (using a different alphabet). Suppose we know that U is a solution tableau (shape λ) to OPAC-024 for G in S_a and V is a solution tableau (shape ρ) for H in S_b (again, different alphabet). Then let μ be a partition larger than λ and ρ with $|\mu| = a+b$. Construct a tableau X of shape μ as follows. Let Q be a SYT of shape ρ such that the lattice word of Q fits μ/λ (and so is counted by the Littlewood-Richardson coefficient). The portion of X in λ is U. The portion of X in μ/λ is the Schensted word corresponding to the pair (V, Q) (see [1] for details; see also [3]). Of course, this idea may be extended to longer direct products.

Example 6. Applying Example 5 to Example 2 and using the fact that *jeu de taquin* preserves the descent set (see [2, Ch. 7 Appendix 1]), it follows that the tableaux X can be chosen to be those tableaux whose maj in the two portions of the tableau (λ and μ/λ) are divisible by a and b, respectively. Care must be taken that the alphabets

in each portion are $[a]$ and $[b]$ respectively, even though the second alphabet is larger than the first.

We note in passing that while it is easy to show that if H is conjugate to a subgroup of G, then $K_{\lambda,G} \leq K_{\lambda,H}$ for all λ the converse is not true. In fact, S_6 contains two non-conjugate Klein 4-groups each of which has three elements of type $2^2 1^2$ and one element of type 1^6.

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On the multiplication table of Jack polynomials

Submitted by ***Per Alexandersson*** *and* ***Valentin Féray***

Let α be a positive real parameter and consider the Jack polynomials $P_\lambda^{(\alpha)}$, indexed by partitions λ. Jack polynomials are standard deformations of Schur functions, which can be defined using either a scalar product or differential operators. They are a degenerate case of the celebrated Macdonald polynomials. For background, we refer to [4] and [10], from which we borrow our notation.

A natural question is how these polynomials multiply, i.e. we want to investigate the coefficients $c_{\mu,\nu}^\lambda(\alpha)$ defined by

$$P_\mu^{(\alpha)} P_\nu^{(\alpha)} = \sum_{\substack{\lambda, \\ |\lambda|=|\mu|+|\nu|}} c_{\mu,\nu}^\lambda(\alpha) P_\lambda^{(\alpha)}.$$

To state our first open problem, we need to introduce two α-deformations of the hook products, H_λ and H'_λ as

$$H_\lambda := \prod_{s\in\lambda} (\alpha a_\lambda(s) + l_\lambda(s) + 1), \quad H'_\lambda := \prod_{s\in\lambda} (\alpha a_\lambda(s) + l_\lambda(s) + \alpha),$$

where $a_\lambda(s)$ and $l_\lambda(s)$ are respectively the arm and the leg lengths of box s (H_λ and H'_λ are denoted $c_\lambda(\alpha)$ and $c'_\lambda(\alpha)$, respectively, in [4]). The following conjecture was stated by Stanley in 1989.

OPAC-026. *([10, Conjecture 8.3]) Prove that, for all partitions λ, μ, ν, the quantity $\widetilde{c}_{\mu,\nu}^\lambda(\alpha) := H'_\lambda H_\mu H_\nu c_{\mu,\nu}^\lambda(\alpha)$ is a polynomial in α with* ***nonnegative*** *integer coefficients.*

Here are some examples of these numbers (taken from [10]; Stanley credits Hanlon for the computations):

$$\widetilde{c}_{31,21}^{421}(\alpha) = 8\alpha^5(9 + 97\alpha + 294\alpha^2 + 321\alpha^3 + 131\alpha^4 + 12\alpha^5);$$
$$\widetilde{c}_{211,211}^{22211} = 284\alpha^5(1+\alpha)^3(4+\alpha)(5+\alpha).$$

When Stanley formulated this problem, it was not even clear whether it is a polynomial in α; (that it is a rational function in α is easy). This polynomiality property, and the integrality of the coefficients, is however a consequence of Knop-Sahi's combinatorial description of Jack polynomials [3]. The still open part of the problem is therefore the nonnegativity.

For $\alpha = 1$, we have $P_\lambda^{(1)} = s_\lambda$, where s_λ is the Schur function associated to λ. The quantities $c_{\mu,\nu}^\lambda(1)$ are therefore the celebrated *Littlewood Richardson coefficients*, which are nonnegative.

For $\alpha = 2$, the Jack polynomials $P_\alpha^{(2)}$ correspond, up to a multiplicative constant, to the so-called *zonal polynomials*. The latter appear in the theory of the Gelfand pair (S_{2n}, H_n) (S_{2n} is the symmetric group on $2n$ elements and H_n its hyperoctahedral subgroup, see [4, Section 7.2] for details). This algebraic interpretation implies the nonnegativity of $c_{\mu,\nu}^\lambda(2)$ [4, VII, (2.28)].

The case $\alpha = 1/2$ follows from $\alpha = 2$ by duality. As far as we are aware of, the question of the nonnegativity of $c_{\mu,\nu}^\lambda(\alpha)$ is open for other values of α. Note also that the nonnegativity for any fixed value $\alpha > 0$ is weaker than the nonnegativity of the coefficients as polynomial in α.

A generalization. As is standard in mathematics, to solve an open problem, it might be useful to generalize it. To this end, we consider the so-called *shifted Jack symmetric functions* or *Jack interpolation polynomials*, denoted $P_\lambda^\#$. These are non-homogeneous "shifted symmetric" functions, whose top homogeneous component is $P_\lambda^{(\alpha)}$. We refer to [7] for a definition of these objects (see also [8, 2]).

We consider the multiplication table of shifted Jack symmetric functions (they form a basis of the ring of shifted symmetric functions):

$$P_\mu^\# P_\nu^\# = \sum_{\substack{\lambda, \\ |\lambda| \leq |\mu| + |\nu|}} d_{\mu,\nu}^\lambda(\alpha) P_\lambda^\# .$$

As before, we renormalize by defining $\widetilde{d}_{\mu,\nu}^\lambda(\alpha) := H'_\lambda H_\mu H_\nu d_{\mu,\nu}^\lambda(\alpha)$.

Here are some examples:

$$\widetilde{d}_{31,21}^{421} = 8\alpha^5(9 + 97\alpha + 294\alpha^2 + 321\alpha^3 + 131\alpha^4 + 12\alpha^5);$$
$$\widetilde{d}_{41,41}^{541} = 144\alpha^8(1+\alpha)^2(2+\alpha)(2+3\alpha)(1+4\alpha)(18 + 149\alpha + 238\alpha^2 + 120\alpha^3).$$

More data data is given **here** (where $\mathrm{JackStructureConstants}[\mu,\nu,\lambda] = \widetilde{d}^{\lambda}_{\mu,\nu}(\alpha)$).

The attentive reader might have noticed that $\widetilde{d}^{421}_{31,21} = \widetilde{c}^{421}_{31,21}$. This is not a coincidence: whenever $|\lambda| = |\mu| + |\nu|$, we have $d^{\lambda}_{\mu,\nu}(\alpha) = c^{\lambda}_{\mu,\nu}(\alpha)$ (this follows from the top component of $P^{\#}_{\lambda}$ being $P^{(\alpha)}_{\lambda}$). However unlike the c coefficients, the d coefficients are also defined when $|\lambda| < |\mu| + |\nu|$.

From the data, we propose the following generalization of OPAC-026, strengthening a conjecture of Sahi [9, Conjecture 6].

OPAC-027. *Show that, for any λ, μ, ν with $|\lambda| \leq |\mu| + |\nu|$, the quantity $\alpha^{|\mu|+|\nu|-|\lambda|-2}\widetilde{d}^{\lambda}_{\mu,\nu}(\alpha)$ is a polynomial in α with **nonnegative** coefficients.*

Again, one can prove the polynomiality in α [1, Section 5], so that the still open part is the nonnegativity of the coefficients.

Why is this generalized conjecture interesting? Sahi [9] has established some recurrence relations for $d^{\lambda}_{\mu,\nu}(\alpha)$ (see [1, Section 6]):

Proposition. *Let $\mu, \nu \subseteq \lambda$ (otherwise, the corresponding coefficient is zero). Then*

$$d^{\lambda}_{\mu,\nu}(\alpha) = \frac{1}{|\lambda| - |\nu|}\left(\sum_{\nu \to \nu^+} \psi'_{\nu^+/\nu} d^{\lambda}_{\mu,\nu^+}(\alpha) - \sum_{\lambda^- \to \lambda} \psi'_{\lambda/\lambda^-} d^{\lambda^-}_{\mu,\nu}(\alpha)\right)$$

where the first sum is taken over all possible ways to add one box to the diagram ν, and the second sum is over all ways to remove one box from λ. Here, $\psi'_{\lambda/\mu} = \psi_{\lambda'/\mu'}$, where $\psi_{\lambda'/\mu'}$ is the coefficient appearing in the Pieri rule for Jack polynomials [4, VI, (10.11)].

The recursion, with the initial conditions $d^{\lambda}_{\mu,\lambda}(\alpha) = P^{\#}_{\mu}(\lambda)$, determines uniquely all coefficients $d^{\lambda}_{\mu,\nu}(\alpha)$, hence in particular the coefficients $c^{\lambda}_{\mu,\nu}(\alpha)$. On the other hand, we are not aware of such recursions involving only the coefficients $c^{\lambda}_{\mu,\nu}(\alpha)$. Hence generalizing Stanley's conjecture gives us an extra tool to attack it.

OPAC-028. *Find some manifestly positive expression in α, e.g. as a weighted enumeration of a family of tableaux depending on λ, μ, ν, which satisfies the above initial conditions and recursion.*

Such a strategy has been performed successfully in the case $\alpha = 1$ (in the more general context of factorial Schur functions), see [6, 5]. If you can generalize this approach to any α, you'll solve a 30-year old conjecture of Stanley, considered by Sahi as "perhaps the most important outstanding problem regarding [Jack] polynomials" [9]…

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Two q,t-symmetry problems in symmetric function theory

Submitted by ***Maria Monks Gillespie***

There are many natural combinatorial problems yet to be solved in the study of two-parameter symmetric functions such as Macdonald polynomials. We describe two of them here, both of which ask to explain the symmetry between q and t exhibited by certain combinatorially defined polynomials in q and t. For the first, a good general reference for the notions involved is [2], and for the second, [4].

Diagonal coinvariants and parking functions

Consider the diagonal action of the symmetric group S_n on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, in which permutations act simultaneously on the two sets of variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. The *diagonal coinvariant ring* R_n is defined as

$$R_n := \mathbb{Q}[X, Y]/J_n$$

where J_n is the ideal generated by all the positive degree invariants under the S_n-action.

The *Frobenius characteristic* of a doubly graded S_n-module R is a two-parameter symmetric function that captures its representation-theoretic information. It is defined as

$$\mathrm{Frob}_{q,t}(R; z_1, z_2, \ldots) := \sum_{i,j,\lambda} c_\lambda^{i,j} s_\lambda(Z) q^i t^j$$

where $c_\lambda^{i,j}$ is the number of copies of the irreducible S_n-module V_λ appearing in the degree i, j component of R, and where $s_\lambda(Z)$ is the Schur function corresponding to λ in the variables $z_1, z_2, \ldots$.

The recently proved *Shuffle Theorem* [1] states that

$$(*) \qquad \mathrm{Frob}_{q,t}(R_n; Z) = \sum_{\sigma \in \mathcal{WP}_n} q^{\mathrm{area}(\sigma)} t^{\mathrm{dinv}(\sigma)} Z^\sigma$$

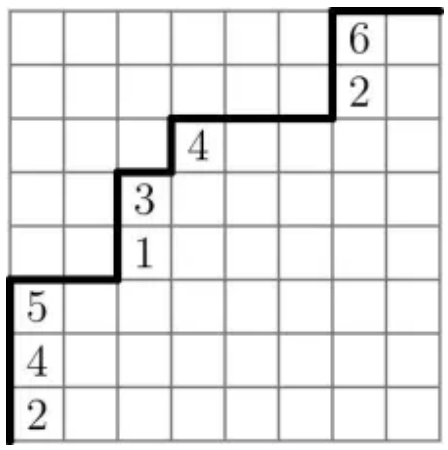
where the terms in the summation are defined as follows:

- $\mathcal{WP}_n$ is the set of all word parking functions of height n, defined as a pair (D, w) where D is a Dyck path from $(0, 0)$ to (n, n), and w is a positive integer labeling of

each grid square to the right of a vertical step in D, such that the labels in each column are increasing from bottom to top.

- $\mathrm{area}(\sigma)$ is the number of grid squares whose interiors lie strictly between the diagonal and the Dyck path.
- $\mathrm{dinv}(\sigma)$ is the number of pairs (i, j) of labels with $i < j$ such that either (a) i and j lie on the same diagonal line $x + y = k$ with i below j, or (b) i is on the diagonal $x + y = k$ and j is on $x + y = k + 1$ for some k, with i above j.
- $Z^\sigma := \prod_i z_i^{m_i(\sigma)}$ where $m_i(\sigma)$ is the number of times the label i appears in σ.

For example, if σ is the word parking function drawn below, we have $\mathrm{area}(\sigma) = 9$, $\mathrm{dinv}(\sigma) = 5$, and $Z^\sigma = z_1 z_2^2 z_3 z_4^2 z_5 z_6$:



Interestingly, the left hand side of equation (*) must be symmetric in q and t, because as a ring R_n is symmetric in the two sets of variables X and Y, which determine the double grading. However, there is not an obvious combinatorial explanation for why the very different statistics dinv and area on the right hand side of (*) should exhibit such a symmetry.

OPAC-029. *Give a combinatorial proof of the q, t-symmetry of the summation in equation (*), by finding a bijection on word parking functions that interchanges dinv and area.*

Let us use R_n^ε to denote the subspace of *antisymmetric elements* of R_n. Some recent progress has been made towards a combinatorial proof of the q, t-symmetry of the Hilbert series of R_n^ε, which is the coefficient of $s_{(1^n)}$ in the Frobenius series of R_n. The Hilbert series of R_n^ε is given by the q, t-*Catalan number*:

$$C_n(q, t) := \sum_{\sigma \in \mathcal{P}_n} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)}$$

where $\mathcal{P}_n$ is the set of all word parking functions whose labels are exactly $1, 2, \ldots, n$ increasing from bottom to top. The paper [6] gives a possible combinatorial approach to the symmetry of $C_n(q, t)$ and a proof in certain special cases.

Garsia-Haiman modules and Macdonald symmetry

The S_n-modules R_n were introduced in the study of Macdonald polynomials because they have important quotients called the *Garsia-Haiman modules*. Given a partition μ, the module R_μ is defined as

$$R_\mu := \mathbb{Q}[X, Y]/J_\mu$$

where J_μ is a larger set of polynomials than J_n but is still invariant under the diagonal action of S_n. (See [4] for more details.)

The Frobenius characteristic of R_μ is the *transformed Macdonald polynomial* $\widetilde{H}_\mu(Z; q, t)$, which was shown in [3] to exhibit the following combinatorial formula:

$$\widetilde{H}_\mu(Z; q, t) = \sum_{\sigma \in \mathcal{F}(\mu)} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} Z^\sigma$$

Here, we have:

- The set $\mathcal{F}(\mu)$ is the set of all *fillings* σ of the Young diagram of shape μ in which each square is filled with a positive integer (with no other restrictions on the entries).
- A *descent* of σ is an entry d that is strictly larger than the entry just below it, and we define $\mathrm{leg}(d)$ to be the number of entries weakly above d in its column. Then $\mathrm{maj}(\sigma) := \sum_d \mathrm{leg}(d)$ where the sum is over all descents d of σ.
- A *relative inversion* of σ is a pair (u, v) of entries in the same row with u to the left of v, such that if b is the entry directly below u (or $b = 0$ if no such entry exists), either:
  - $u < v$ and b is between u and v in size, in particular $u \leq b < v$, or
  - $u > v$ and b is not between u and v in size, in particular either $b < v < u$ or $v < u \leq b$.

  Then $\mathrm{inv}(\sigma)$ is the number of relative inversions of σ.
- $Z^\sigma := \prod_i z^{m_i(\sigma)}$ where $m_i(\sigma)$ is the number of times the label i appears in σ.

If σ is the example filling of shape $\mu = (4, 4, 2, 1)$ drawn below, we have $\mathrm{maj}(\sigma) = 5$, $\mathrm{inv}(\sigma) = 4$, and $Z^\sigma = z_1^2 z_2^2 z_3^2 z_4^2 z_5^2 z_6$:

1			
5	4		
3	2	2	5
3	1	4	6

Define μ^* to be the conjugate of a given partition μ, formed by reflecting its Young diagram about the diagonal. Due to the definition of the ideals J_μ, the Macdonald polynomials exhibit 'conjugate symmetry' in q and t in the sense that:

$$(**) \qquad \widetilde{H}_\mu(Z;q,t) = \widetilde{H}_{\mu^*}(Z;t,q)$$

OPAC-030. *Give a combinatorial proof of the conjugate q,t-symmetry of the summation in equation (**).*

In [5], the author found a bijection between fillings that conjugates the partition and switches inv and maj for hook shapes μ, as well as for several more families of shapes in the specialization at $q=0$ (in other words, when restricting to those fillings σ which have $\mathrm{inv}(\sigma)=0$). However, a complete combinatorial explanation of the symmetry between the statistics remains elusive.

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Coinvariants and harmonics

Submitted by ***Mike Zabrocki***

Parts of this open problem are well studied and the results are well known, other aspects and variations have been barely explored.

Start with the polynomial ring in k sets of n commuting variables $x_{i,j}$ and ℓ sets of n anti-commuting variables $\theta_{i,j'}$. That is there are variables $x_{i,j}$ and $\theta_{i,j'}$ with $1 \leq i \leq n$, $1 \leq j \leq k$, $1 \leq j' \leq \ell$ satisfying the relations

$$x_{a,b}x_{c,d} = x_{c,d}x_{a,b} \qquad x_{a,b}\theta_{c',d'} = \theta_{c',d'}x_{a,b} \qquad \theta_{a',b'}\theta_{c',d'} = -\theta_{c',d'}\theta_{a',b'}$$

for all $1 \leq a, c, a', c' \leq n$, $1 \leq b, d \leq k$ and $1 \leq b', d' \leq \ell$. That is we are looking at the ring of polynomials in these variables and we will denote this polynomial ring as

$$\mathbb{C}[X_{n\times k}; \Theta_{n\times \ell}] := \mathbb{C}[x_{i,j}, \theta_{i,j'} : 1 \leq i \leq n, 1 \leq j \leq k, 1 \leq j' \leq \ell].$$

Now let W be a group which acts on the first index of the variables. That is, for each $w \in W$, there exists coefficients $a_{i,j}$ and $a'_{i,j}$ such that

$$w(x_{i,b}) = \sum_{j=1}^{n} a_{i,j}x_{j,b} \qquad w(\theta_{i,b'}) = \sum_{j=1}^{n} a'_{i,j}\theta_{j,b'}$$

for all $1 \leq b \leq k$ and $1 \leq b' \leq \ell$ and this action is extended to act on the monomials in the variables. An invariant is a polynomial $f \in \mathbb{C}[X_{n\times k}; \Theta_{n\times \ell}]$ such that $wf = f$ for all $w \in W$. The W-invariant polynomials are closed under multiplication and addition and form a subring of $\mathbb{C}[X_{n\times k}; \Theta_{n\times \ell}]$.

The *coinvariant ring* (and W-module) is the quotient of the ideal generated by the invariants with no constant term. That is, it is the space defined as

$$C^W_{k,\ell} := \mathbb{C}[X_{n\times k}; \Theta_{n\times \ell}]/I^+_W$$

where I^+_W is the ideal generated by the invariants of the action of W that have no constant term (if the constant term is included in this ideal then the ideal includes the whole ring).

OPAC-031. *Describe the structure of $C^W_{k,\ell}$ as completely as possible (dimension, ring structure, decomposition into W-irreducibles, Gröbner basis of I^+_W, resolutions, geometric interpretations, etc.).*

There is an isomorphic formulation of this construction in terms of polynomials which are killed by all symmetric polynomials in differential operators with non-constant term. This space is usually referred to as 'harmonics' and certain aspects of this space are easier to understand or calculate through the isomorphism between the coinvariants and harmonics (see [4]).

The most interesting case for me is when $W = S_n$ and I provide a table below summarizing the dimensions for small k and ℓ as sequences in n referring to the OEIS sequence number (these are all conjectural based on only a few values except for $(k, \ell) \in \{(0,0), (1,0), (2,0), (0,1)\}$ where the result has been proven). The first 7 terms of $(k, \ell) = (2, 2)$ entry are 1, 1, 5, 45, 597, 10541, 233157. For $(k, \ell) = (3, 1)$, the first 7 entries are 1, 1, 5, 50, 785, 17072, 478205 and for $(k, \ell) = (3, 2)$ the first 7 entries are 1, 1, 6, 74, 1440, 38912, 1356096. None of these three sequences are currently in the OEIS.

(Mostly conjectured) dimensions of $C_{k,\ell}^{S_n}$ for small k and ℓ.

	$\ell = 0$	$\ell = 1$	$\ell = 2$
$k = 0$	1	2^n – A000079	$\binom{2n+1}{n+1}$ – A001700
$k = 1$	$n!$ – A000079	A000670	$2^{n-1}n!$ – A002866
$k = 2$	$(n+1)^{n-1}$ – A000272	A201595	?
$k = 3$	$(n+1)^{n-2}2^n$ – A127670	?	?

A list of references that touches on all the special cases of this problem that people have looked at before would be quite long. The best known special case is $k = 1, \ell = 0$ and then a description follows from work of Chevalley-Sheppard-Todd [2, 11] which says that the dimension of $C_{1,0}^W$ is equal to the order of W if and only if W is a group generated by pseudo-reflections. In that case, $C_{1,0}^W$ is isomorphic as a W-module to the regular representation of W. This is the level of understanding that we would like to have of $C_{k,\ell}^W$ in all cases.

Some computational experiments leads us to believe that the dimension of $C_{1,1}^W$ is equal to the order of the **Coxeter complex** of W, but only 'most of the time.' Brendon Rhoades and Josh Swanson [10] had made this observation but then found that the dimension of $C_{1,1}^{F_4}$ is not equal to the order of the Coxeter complex. I tried computing some of these and found that for small values this was true in types $A_{n'}$ $B_{n'}$ $D_{n'}$ $I_2(m)$ but then I wasn't able to compute the coinvariants with $W = E_6$ using Macaulay2 without running out of memory so the data may not be in any way robust enough to call this a conjecture. Pinning down the precise statement in this case would be a really nice extension to the work of Chevalley-Sheppard-Todd. Josh Swanson [12] gave a description of the alternating part of $C_{1,1}^W$.

In the case of $k = 2$ and $\ell = 0$ and $W = S_{n'}$ Mark Haiman [8] conjectured in the early 90's that the dimension of $C_{2,0}^{S_n}$ is $(n+1)^{n-1}$ and the multiplicity of the alternating

representation is equal to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. In that same paper he also conjectured the dimension and multiplicity of the alternating representation for $k = 3$ and $\ell = 0$. The dimensions for the $k = 2$ case were proven in 2000 [9] and Iain Gordon [5] then proved analogous results for W a finite Coxeter group. In the following years an extension to a combinatorial formula for the graded Frobenius image of the character of $C_{2,0}^{S_n}$ became known as 'The Shuffle Conjecture' [6] and this formula was proven in 2016 by Carlsson and Mellit [1]. The Schur expansion of the expression in The Shuffle Conjecture is not known and I stated this open problem in a way that indicates there is room to still explore this case.

Haglund, Remmel and Wilson [7] proposed an extension to the combinatorial and symmetric function formulae that appear in The Shuffle Conjecture and named it 'The Delta Conjecture.' The idea for this open problem comes from a conjecture that I posted [14] in February 2019 that their proposed expression is encoded in the graded character of $C_{2,1}^{S_n}$.

François Bergeron [3] has been looking a module that is conjecturally isomorphic to the module $C_{k,0}^{W}$. He encoded the character/Frobenius character in a multivariate expression and has been able to compute it for $W = S_n$ up to $n = 6$ for any k (and I used his symmetric function expressions to calculate the data for the above table).

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Isomorphisms of zonotopal algebras

Submitted by ***Gleb Nenashev***

Zonotopal algebras were defined for hyperplane arrangements independently by F. Ardila and A. Postnikov in [2] and O. Holtz and A. Ron in [3]. Prior to these works A. Postnikov and B. Shapiro defined the family of graphical algebras; see [5, 6]. Here we present definitions and the problem only for the graphical case. For the case of unimodular zonotopal algebras and general zonotopal algebras, all definitions, results, and the conjecture can be extended; see more details in [4].

Let G be a graph on $n+1$ vertices. We index its vertices with numbers 0 to n, i.e., $V(G) = \{0\} \cup [n]$. For an integer k, consider the ideal $\mathcal{J}_G^{(k)}$ in the ring $\mathbb{R}[x_1, \cdots, x_n]$ generated by

$$p_I^{(k)} := \left(\sum_{i \in I} x_i \right)^{d_I + k}, \qquad \varnothing \neq I \subseteq [n],$$

where d_I is the number of edges of G connecting I and $V(G) \setminus I$. Define $C_G^{(k)}$ to be the quotient algebra $\mathcal{C}_G^{(k)} := \mathbb{R}[x_1, \cdots, x_n] / \mathcal{J}_G^{(k)}$.

These algebras are independent on the choice of the root (i.e., which vertex is labeled 0). There are 3 main cases of these algebras, namely:

- k = 1: *External* zonotopal algebra;
- k = 0: *Central* zonotopal algebra;
- k = −1: *Internal* zonotopal algebra.

All these 3 graphical zonotopal algebras are graded. Their total dimensions (as linear spaces) and their Hilbert series both admit good combinatorial interpretations.

Theorem. *(cf. [2, 3, 5, 6]) Given a connected graph G, the Hilbert series of zonotopal algebras are given by*

- $H_{\mathcal{C}_G^{(1)}} = q^{e(G)-v(G)+1}\, T_G(1+q, 1+q)$;
- $H_{\mathcal{C}_G^{(0)}} = q^{e(G)-v(G)+1}\, T_G(1, 1+q)$;

- $H_{\mathcal{C}_G^{(-1)}} = q^{e(G)-v(G)+1}\, T_G(0, 1+q)$,

where T_G is the Tutte polynomial of G.

Corollary. *(cf. [5]) For a connected graph G,*

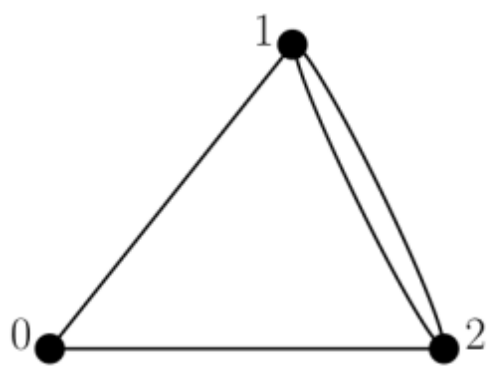
- *the dimension of the external zonotopal algebra $\mathcal{C}_G^{(1)}$ is the number of forests of G;*
- *the dimension of the central zonotopal algebra $\mathcal{C}_G^{(0)}$ is the number of trees of G.*

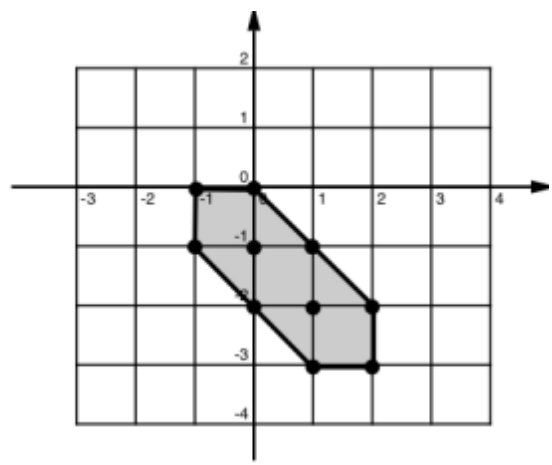
In fact, the above theorem applies to any vector configuration. Furthermore, if a configuration is totally unimodular (that is, if any minor of the corresponding matrix is ± 1 or 0), then the total dimensions of these three algebras are the number of lattice points, the volume, and the number of internal lattice points of the corresponding zonotope.

Example. Let G be a graph on the vertex set $\{0, 1, 2\}$ with 4 edges: $(0,1)$, $(0,2)$, $(1,2)$, and $(1,2)$. Then $T_G(x,y) = x + y + x^2 + xy + y^2$ is the Tutte polynomial of G. Its corresponding zonotope is defined as the following Minkowski sum of edges:

$$Z_G := [(0,0);(-1,0)]+[(0,0);(0,-1)]+[(0,0);(1,-1)]+[(0,0);(1,-1)],$$

The graph G and its corresponding zonotope Z_G are depicted below:





In this case, the zonotopal ideals are

$$\mathcal{J}_G^{(k)} = \langle x_1^{3+k}, x_2^{3+k}, (x_1+x_2)^k \rangle$$

and

- $H_{\mathcal{C}_G^{(1)}}(q) = 1 + 2q + 3q^2 + 3q^3 + q^5$, and the total dimension of $\mathcal{C}_G^{(1)}$ is 10;

- $H_{\mathcal{C}_G^{(0)}}(q) = 1 + 2q + 2q^2$, and the total dimension of $\mathcal{C}_G^{(0)}$ is 5;
- $H_{\mathcal{C}_G^{(-1)}}(q) = 1 + q$, and the total dimension of $\mathcal{C}_G^{(0)}$ is 2.

The graph G has exactly 10 forests and 5 trees. Furthermore, the number of lattice points, the area, and the number of internal lattice points of Z_G equal 10, 5, and 2, respectively.

The main result of [4] is the classification of the external zonotopal algebras up to isomorphism. In the graphical case the result gives

Theorem. *(cf. [4]) Let G_1 and G_2 be two graphs. Then the following are equivalent:*

- *$\mathcal{C}_{G_1}^{(1)}$ and $\mathcal{C}_{G_2}^{(1)}$ are isomorphic as non-graded algebras;*
- *$\mathcal{C}_{G_1}^{(1)}$ and $\mathcal{C}_{G_2}^{(1)}$ are isomorphic as graded algebras;*
- *the graphical matroids M_{G_1} and M_{G_2} are isomorphic.*

The cases of central and internal zonotopal algebras are still open.

OPAC-032. *Classify central and internal zonotopal algebras up to isomorphism.*

In the case of the central zonotopal algebra, we have the following specific conjecture, which appears in [4]:

Conjecture. *Given two connected graphs G_1 and G_2, the central zonotopal algebras of G_1 and G_2 are isomorphic if and only if the bridge-free matroids of these graphs are isomorphic (where by bridge-free matroid we mean the matroid obtained after removing all the bridges).*

The proof of the classification of external zonotopal algebras is based on an alternative definition of the external zonotopal algebra, where we consider the algebra as a subalgebra of a square-free algebra; see [6]. There is an analogous alternative definition for the central algebra as well (see [1, Section 4]).

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Banff, Louise, and class P quivers

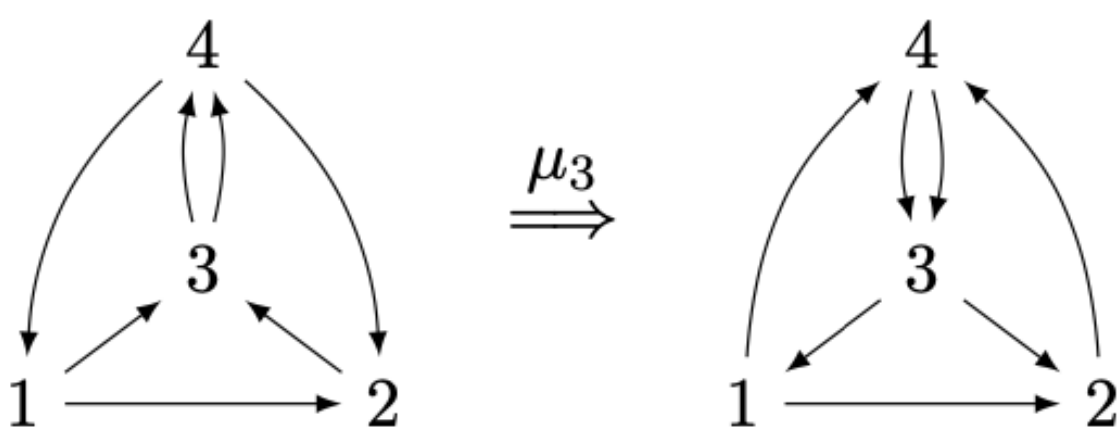
Submitted by ***Eric Bucher*** *and* ***John Machacek***

Cluster algebras are commutative rings defined by Fomin and Zelevinsky [1] with a distinguished set of generators determined by a combinatorial process known as *mutation*. These algebras are known to be connected with various areas of mathematics and physics. Hence, the theory of cluster algebras has many directions and open problems. Here, we will focus on some purely combinatorial open problems in cluster algebra theory with the hope of giving researchers in combinatorics an avenue into cluster algebras. Also, we think these problems will interest cluster algebra researchers and give them ideas for combinatorial applications of their work.

The problems here will make no mention of the ring structure nor the generators. This is not to say algebra is not still present. Solutions to the problems mentioned would have an immediate impact on the corresponding cluster algebras. Furthermore, even though the problems are combinatorial in statement it is certainly possible solutions could be obtained by other means.

A *quiver*, Q, is a directed graph with no loops or 2-cycles. *Mutation* at a vertex k is a process which produces a new quiver, denoted $\mu_k(Q)$, by using the following three steps in order:

1. For each pair of arrows $i \to k, k \to j$ add an arrow $i \to j$.
2. Reverse the direction of all arrows incident to G.
3. Delete a maximal collection of disjoint 2-cycles.



A *bi-infinite path* in a quiver Q is a sequence $(i_a)_{a\in\mathbb{Z}}$ of vertices such that $i_a \to i_{a+1}$ is an arrow for each $a \in \mathbb{Z}$. A pair of vertices (i, j) is a *covering pair* if $i \to j$ is an arrow

that is not part of any bi-infinite path. Muller's class of *Banff* quivers [2] is defined as the smallest class of quivers such that

- any acyclic quiver is Banff,
- any quiver mutation equivalent to a Banff quiver is Banff,
- and any quiver Q with a covering pair (i, j) where both $Q \setminus \{i\}$ and $Q \setminus \{j\}$ are Banff is a Banff quiver.

Here a quiver is *acyclic* if it contains no directed cycles. Banff quivers are of importance because the cluster algebras they define are *locally acyclic* which means they enjoy many nice properties including being finitely-generated, integrally closed, and equal to their upper cluster algebras [2] [3]. Also, Banff quivers admit reddening sequences [4]. A *reddening sequence* is a desirable sequence of mutations defined by Keller (see [5] for a survey). Speyer and Lam have modified the Banff property and defined the class of *Louise* quivers [6] as the smallest class of quivers such that

- any acyclic quiver is Louise,
- any quiver mutation equivalent to a Louise quiver is Louise,
- and any quiver Q with a covering pair (i, j) where each of $Q \setminus \{i\}$, $Q \setminus \{j\}$, and $Q \setminus \{i, j\}$ is Louise is a Louise quiver.

The definition for Banff and Louise quivers are very similar. The only difference is to be Louise we must additionally check $Q \setminus \{i, j\}$ for a covering pair (i, j). This extra condition allows for Mayer-Vietoris sequences to be used in computing the cohomology of the corresponding cluster varieties.

OPAC-033. *Find an example of a Banff quiver which is not Louise or prove no such quiver exists.*

If the class of Banff quivers coincides with the class of Louise quivers it would be advantageous to know this since one could then use the full power of Louise quivers while only checking the Banff condition. On the other hand if these two classes of quivers are different it would be interesting to understand what the difference between them is.

In practice, covering pairs involving sources or sinks are often used to show a quiver is Banff, but it is unclear whether one can obtain all Banff quivers this way. Let $\mathcal{B}$ denote the class of Banff quivers and let $\mathcal{B}'$ denote the subclass of Banff quivers which only use covering pairs such that at least one of the two vertices is a source or sink.

OPAC-034. *Find an example of a quiver in $\mathcal{B} \setminus \mathcal{B}'$ or prove no such quiver exists.*

Understanding the difference $\mathcal{B}$ and $\mathcal{B}'$ would be interesting as well as potentially useful in checking the Banff conditions. Additionally, the quivers in $\mathcal{B}'$ have stronger implications, including equality of the quantum cluster algebra with the quantum upper cluster algebra [7]. More problems of this type include relations to Kontsevich and

Soibelman's [8] class $\mathcal{P}$ (which we will not define here). For example, we have the following problem of containment between Banff quivers and the class $\mathcal{P}$.

OPAC-035. *Find an example of a Banff quiver which is not in the class $\mathcal{P}$ or prove no such quiver exists.*

For further discussion of OPAC-034 and OPAC-035 see [4].

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The Dedekind-MacNeille completion of Type B Bruhat order

Submitted by ***Sam Hopkins***

Let P be a poset. The *Dedekind-MacNeille completion* of P is the "smallest" complete lattice L with an embedding $P \subset L$, in the sense that L is a subposet of any other such lattice. (We will exclusively be interested in finite posets and lattices, for which completeness is automatic, so from now on we will drop the adjective complete.) It can be constructed explicitly from P in much the same way that the real numbers can be obtained from the rationals using Dedekind cuts, hence the name.

Consider the case $P = S_n$, the (strong) Bruhat order on the symmetric group. Unlike weak order, Bruhat order is not a lattice, so it makes sense to ask about its Dedekind-MacNeille completion. In [4], Lascoux and Schützenberger describe the Dedekind-MacNeille completion of S_n in terms of alternating sign matrices.

Recall that an *alternating sign matrix (ASM)* is a matrix with entries in $\{-1, 0, +1\}$ such that: every row and every column sums to one; the nonzero entries in every row and column alternate in sign. The number of $n \times n$ ASMs is given by the famous formula

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} \qquad \text{(http://oeis.org/A005130)}$$

conjectured by Mills-Robbins-Rumsey, proved by Zeilberger (and later Kuperberg).

For a $n \times n$ alternating sign matrix $A = (a_{i,j})$, define its *rank matrix* $r(A) = (r_{i,j})$ by $r_{i,j} = \sum_{i'=1}^{i} \sum_{j'=j}^{n} a_{i',j'}$. That is, each entry of the rank matrix records the sum of the entries weakly northeast of that entry in the ASM. Then define a partial order on $n \times n$ ASMs by $A \leq A'$ iff $r(A) \leq r(A')$ entrywise. We denote this poset (which is in fact a distributive lattice) by ASM_n.

Lascoux and Schützenberger [4] showed that ASM_n is the Dedekind-MacNeille completion of S_n. Furthermore, the embedding $S_n \subset \mathrm{ASM}_n$ just corresponds to viewing a permutation matrix as an ASM.

The open problem is to do the same for B_n, the Bruhat order on the Type B Coxeter group, a.k.a. the group of signed permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$, a.k.a. the hyperoctahedral group. More specifically, there is a natural embedding $B_n \subset S_{2n}$ where

we view B_n as the subset of "half-turn symmetric" $2n \times 2n$ permutation matrices, i.e., those invariant under $180°$ rotation. This means that the Dedekind-MacNeille completion of B_n can naturally be viewed as a subset of ASM_{2n}.

OPAC-036. *Describe the Dedekind-MacNeille completion of B_n as a subset of ASM_{2n}.*

For prior work on the Dedekind-MacNeille completion of B_n and related matters like a description of its join irreducible elements, in addition to the work of Lascoux and Schützenberger [4], consult also Geck and Kim [3], Reading [5], Anderson [1], and Armstrong and McKeown [2].

A natural guess would be that the Dedekind-Macneille completion of B_n consists of the half-turn symmetric $2n \times 2n$ ASMs. Certainly the half-turn symmetric ASMs form a sublattice of ASM_{2n}. And the number of such ASMs also has a beautiful product formula

$$\prod_{k=0}^{n-1} \frac{(3k)!(3k+2)!}{((n+k)!)^2} \qquad \text{(http://oeis.org/A059475)}$$

which for $n = 1, 2, 3, 4$ gives $2, 10, 140, 5544$. However, the number of elements in the Dedekind-Macneille completion of B_n gives the sequence $2, 10, 132, 4824$. (This last value 4824 for $n = 4$ was computed by Angela Chen as an undergraduate at the University of Minnesota.) So we see that the Dedekind-Macneille completion of B_n does not consist of all half-turn symmetric $2n \times 2n$ ASMs.

Regarding numerics, this brings us to the next open problem.

OPAC-037. *Give a formula for the number of elements in the Dedekind-MacNeille completion of B_n.*

Since $4824 = 2^3 \times 3^2 \times 67$ has a large prime factor, there may be no nice formula.

Returning to the question of how to characterize the Dedekind-MacNeille completion of B_n inside of ASM_{2n}, we note that it must consist of some *proper subset* of the half-turn symmetric ASMs. For instance,

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ + & 0 & - & 0 & + & 0 \\ 0 & + & 0 & - & 0 & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{pmatrix} \in \mathrm{ASM}_6$$

(where $+$ means $+1$ and $-$ means -1) is not a join of half-turn symmetric permutation matrices, hence not in the Dedekind-MacNeille completion of B_n.

Possibly the following additional requirements, which were proposed by David Speyer in comments on his MathOverflow question [6], suffice. For $1 \leq j, k \leq n$, consider the submatrix of our ASM whose rows are $\{1, 2, \ldots, j\}$ and columns are $\{n-k+1, n-k \ldots, n+k\}$. We require that this submatrix has at most k more $+1$'s than -1's. (The above matrix fails this condition with $j = 2$ and $k = 1$.) We also impose the same condition but with rows and columns switched. It can be shown that the half-turn symmetric $2n \times 2n$ ASMs satisfying these conditions form a lattice inside of ASM_{2n}, which may be the Dedekind-MacNeille completion of B_n.

Of course, it is also natural to wonder about the Dedekind-MacNeille completion of Bruhat order in other types. However, we warn that while the Dedekind-MacNeille completion of Bruhat order in Types A and B is a *distributive* lattice, this is not true for Types D, E and F. (For more on this subtlety, see for instance the discussion of the "dissective" property in [5].) So it might be much harder to understand the Dedekind-MacNeille completion of Bruhat order outside of Types A and B.

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