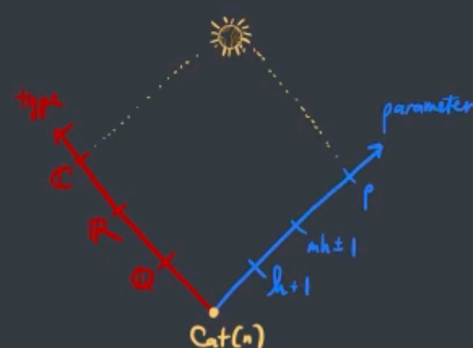
Catalan Combinatorics



OPAC 2022

Pavel Galashin
Thomas Lam
Minh-Tâm Trinh
Nathan Williams

10 YEARS AGO

American Institute of Mathematics

Rational Catalan combinatorics

December 17 to December 21, 2012

at the

American Institute of Mathematics, Palo Alto, California

organized by

Drew Armstrong, Stephen Griffeth, Victor Reiner, and Monica Vazirani

This workshop, sponsored by AIM and the NSF, will be devoted to understanding the interaction between new developments in algebra and combinatorics. In particular, it will focus on combinatorial objects counted by generalizations of Catalan numbers and their interaction with the representation theory of Cherednik algebras.

REF Rational Catalan Combinatorics: An Outline from the AIM workshop, Dec 2012

THIS YEAR

POSITROIDS, KNOTS, AND q,t-CATALAN NUMBERS

PAVEL GALASHIN AND THOMAS LAM

ABSTRACT. We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over $\mathbb{C}$ and point counts over $\mathbb{F}_q$) to Khovanov–Rozansky homology of associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational q,t-Catalan numbers. Via the curious Lefschetz property of cluster varieties, this implies the q,t-symmetry and unimodality properties of rational q,t-Catalan numbers. We show that the q,t-symmetry phenomenon is a manifestation of Koszul duality for category $\mathcal{O}$, and discuss relations with open Richardson varieties and extension groups of Verma modules.

Nathan Williams

Parking analogue of Galashin-Lam?

To: Pavel Galashin

January 21, 2022 at 10:29 AM

Hi Pavel,

```
n=3
R.<q> = QQ[]
W = WeylGroup(['A',n,1])
KL = KazhdanLusztigPolynomial(W, q)
f=KL.R(1,W.from_reduced_word(list(range(n+1))*n))/(q-1)^(2*n)
f==sum([q^i for i in range(n+1)])^(n-1)
```

Best,
Nathan

PROVED February 4, 2022: Galashin-Lam's R-polynomial via Hecke algebra, Opdam's trace formula, and Haglund's Tesler matrix identity.

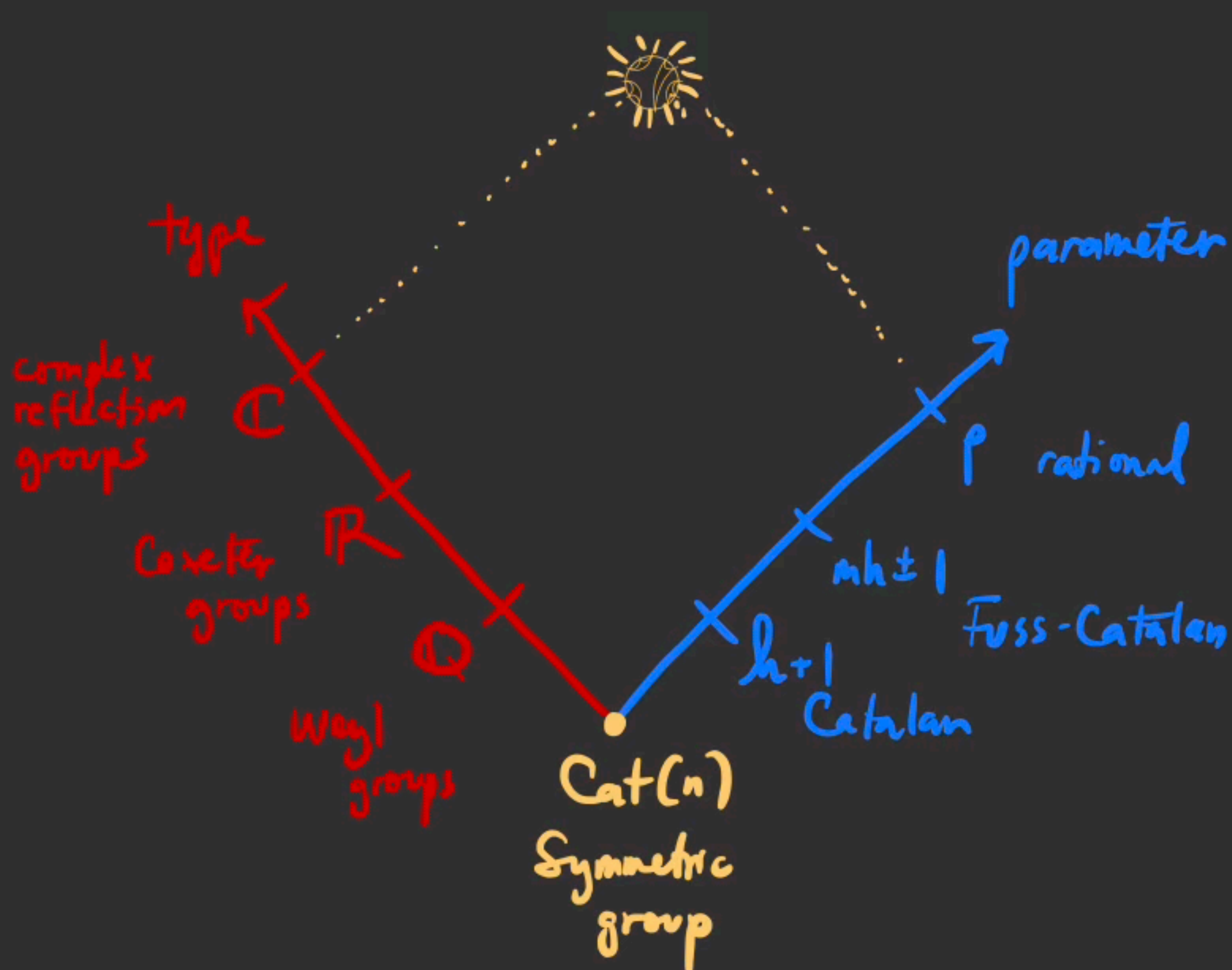
NOT TODAY.

CONTINUED EXPERIMENTS into March, which led to THIS TALK.

We had been looking for these results since the AIM conference

10 YEARS AGO

0. Catalan Numbers



DEF The **Catalan numbers** are the integers

Pak credits Riordan for the name

$$\mathrm{Cat}(n) = \frac{1}{2n+1}\binom{2n+1}{n}$$

EX 1, 1, 2, 5, 14, 42, 132, ...



Cat(n) counts: noncrossing partitions, triangulations, Dyck paths,

etc, etc, etc, etc, etc, ...

REF Pak. "History of Catalan Numbers"
Stanley. "Catalan Numbers"

"<u>THM</u>" (Folklore)

Just about every combinatorial object is Catalan.

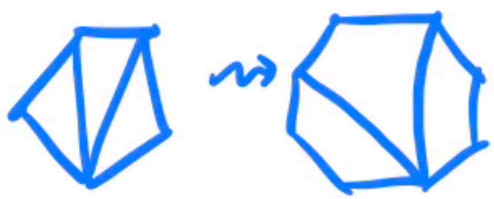
DEF The Catalan numbers are the integers

$$\mathrm{Cat}(n) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{(n+1)+n}\binom{n+1+n}{n}$$

$$= \prod_{i=1}^{n-1} \frac{(n+1)+i}{i+1}$$

$n+1$

PARAMETER

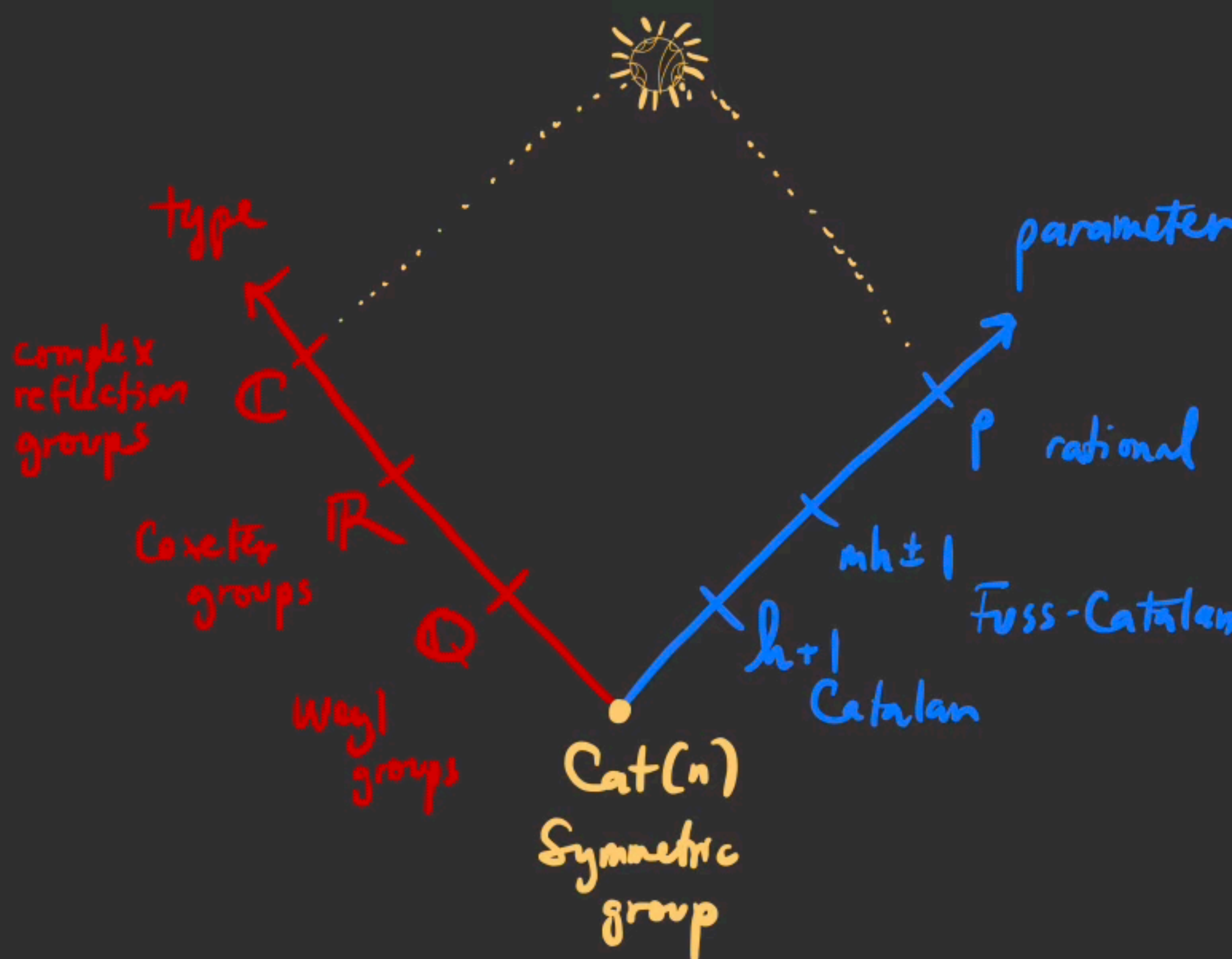


n

TYPE

REF Stanley, "Catalan Numbers"

I. REFLECTION GROUPS



TYPE A

PHILOSOPHY (Tits): "$\mathfrak{S}_n$ is $SL_n(\mathbb{F}_q)$ at $q=1$"

$$|SL_n(\mathbb{F}_q)| = (q-1)^{n-1} q^{\binom{n}{2}} \prod_{i=1}^{n-1} [i+1]_q$$

- Lie group $SL_n(\mathbb{F}_q)$ Tymoczko's talk
- Braid group B_n Gorsky's talk
- Hecke algebra $\mathcal{H}_n$ Mellit's talk
- Affine symmetric group $\widetilde{\mathfrak{S}}_n$ Speyer's talk

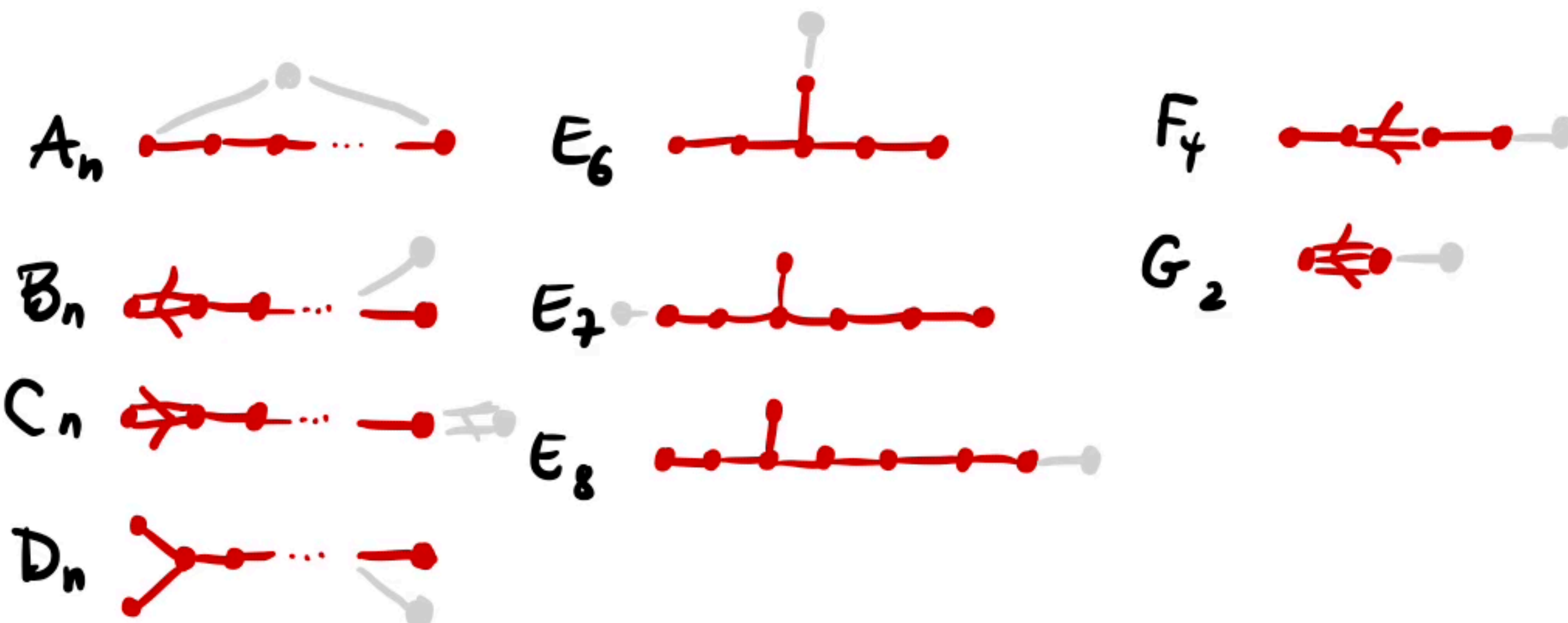
WEYL GROUPS ($\mathbb{Q}$)

- connected reductive group over $\overline{\mathbb{F}}_q$, Frobenius F — G
- Weyl group — $W = N_G(T)/T$ — PHILOSOPHY (Tits): "W is G^F at $q=1$"
- Braid group — $B_W = \pi_1(V^{reg}/W)$
- Hecke algebra — $\mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$
- Affine Weyl group — $\widetilde{W} = W \ltimes Q^\vee$

CLASSIFICATION: WEYL GROUPS ($\mathbb{Q}$)

THM The list of irreducible Weyl groups is:

connected Dynkin diagram



REF Coxeter, "The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$." 1935

COXETER GROUPS ($\mathbb{R}$)

DEF A Coxeter system (W,S) is a group W with presentation $W = \langle s_1, s_2, \dots, s_n \mid (s_i s_j)^{m_{ij}} = id \rangle$

$m_{ij} \in \mathbb{N}$, $m_{ii} = 1$

S is the set of simple reflections.

(Coxeter groups act as reflection groups on $\mathbb{R}^n$ with corresponding hyperplane arrangement $\mathcal{H}_W$)

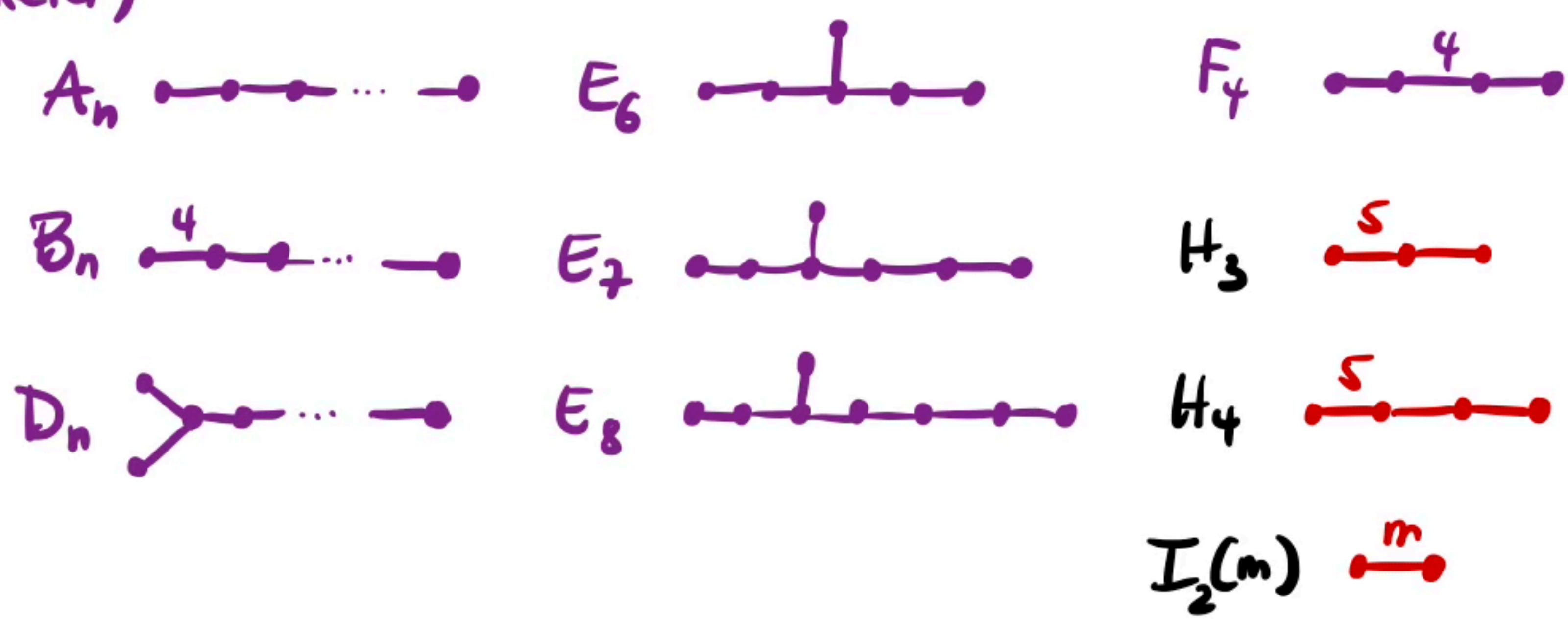
- Braid group $\qquad B_W = \pi_1(V^{reg}/W)$
- Hecke algebra $\qquad \mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$
- Affine Weyl group $\qquad \tilde{W}$
- Lie group

REF Hiller. "Geometry of Coxeter groups"
Humphreys. "Reflection groups and Coxeter groups"
Björner & Brenti, "Combinatorics of Coxeter Groups"

CLASSIFICATION: COXETER GROUPS ($\mathbb{R}$)

THM (Coxeter) The list of finite irreducible Coxeter groups is:

connected Coxeter diagram



REF Coxeter, "The complete enumeration of finite groups of the form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$." 1935

COMPLEX REFLECTION GROUPS ($\mathbb{C}$)

DEF A <u>complex reflection group</u> is a group $W \subseteq GL_n(\mathbb{C})$ generated by complex reflections.

Braid group	$B_W = \pi_1(V^{reg}/W)$
Simple reflections	S
Hecke algebra	$\mathcal{H}_W$ = quotient of $\mathbb{C}[B_W]$
Affine Weyl group	$\widetilde{W}$
Lie group (spetses)	

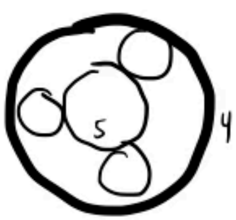
REF Broué, Malle, Rouquier, "On Complex Reflection Groups and their Associated Braid Groups." 1994
Broué, Malle, Rouquier. "Complex reflection groups, braid groups, Hecke algebras". 1998
Shephard, Todd. "Finite Unitary Reflection Groups". 1953

CLASSIFICATION: COMPLEX REFLECTION GROUPS ($\mathbb{C}$)

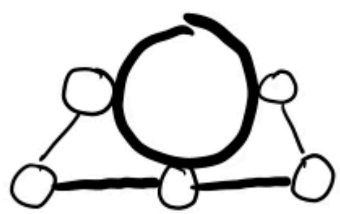
THM (Shephard, Todd) The list of finite irreducible complex reflection groups is:

- $G(m,p,n)$
- G_4
- G_5
- $\vdots$
- G_{37} $= E_8$

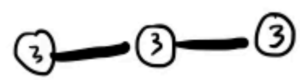
Ex



G_{13}



G_{31}



G_{25}

REF Broué, Malle, Rouquier, "On Complex Reflection Groups and their Associated Braid Groups." 1994

Broué, Malle, Rouquier. "Complex reflection groups, braid groups, Hecke algebras". 1998

Shephard, Todd. "Finite Unitary Reflection Groups". 1953

INVARIANT THEORY AND NUMEROLOGY

W acts on $\mathbb{C}^n = \mathrm{span}_{\mathbb{C}}\{x_1, \dots, x_n\}$, hence on $\mathbb{C}[x_1, \dots, x_n]$.

THM (Chevalley) Let $W \subseteq GL_n(\mathbb{C})$. Then

W is a complex reflection group iff $\mathbb{C}[x_1, \dots, x_n]^W = \mathbb{C}[f_1, \dots, f_n]$.

DEF Let $\deg f_i = d_i$ with $\underbrace{d_1 \leq d_2 \leq \cdots \leq d_n}_{\text{degrees}}$.

($h = d_n$ is the Coxeter number. $e_i = d_i - 1$ are the exponents)

EX $\mathfrak{S}_n \curvearrowright \mathbb{C}^{n-1}$ has invariant polys power sum, elementary, homogeneous, Schur, monomial, forgotten, ...

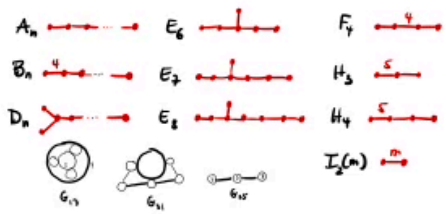
but always $\deg f_i = i+1$.

REF Chevalley. Invariants of finite groups generated by reflections.

THE GOLD STANDARD

$\underbrace{\mathbb{Z}/\mathbb{R}/\mathbb{C}\text{-UNIFORM}}$ definitions & proofs for reflection groups.

"does not appeal to the $\mathbb{Z}/\mathbb{R}/\mathbb{C}$-classification"



Ex $|W| = \prod_{i=1}^{n} d_i$

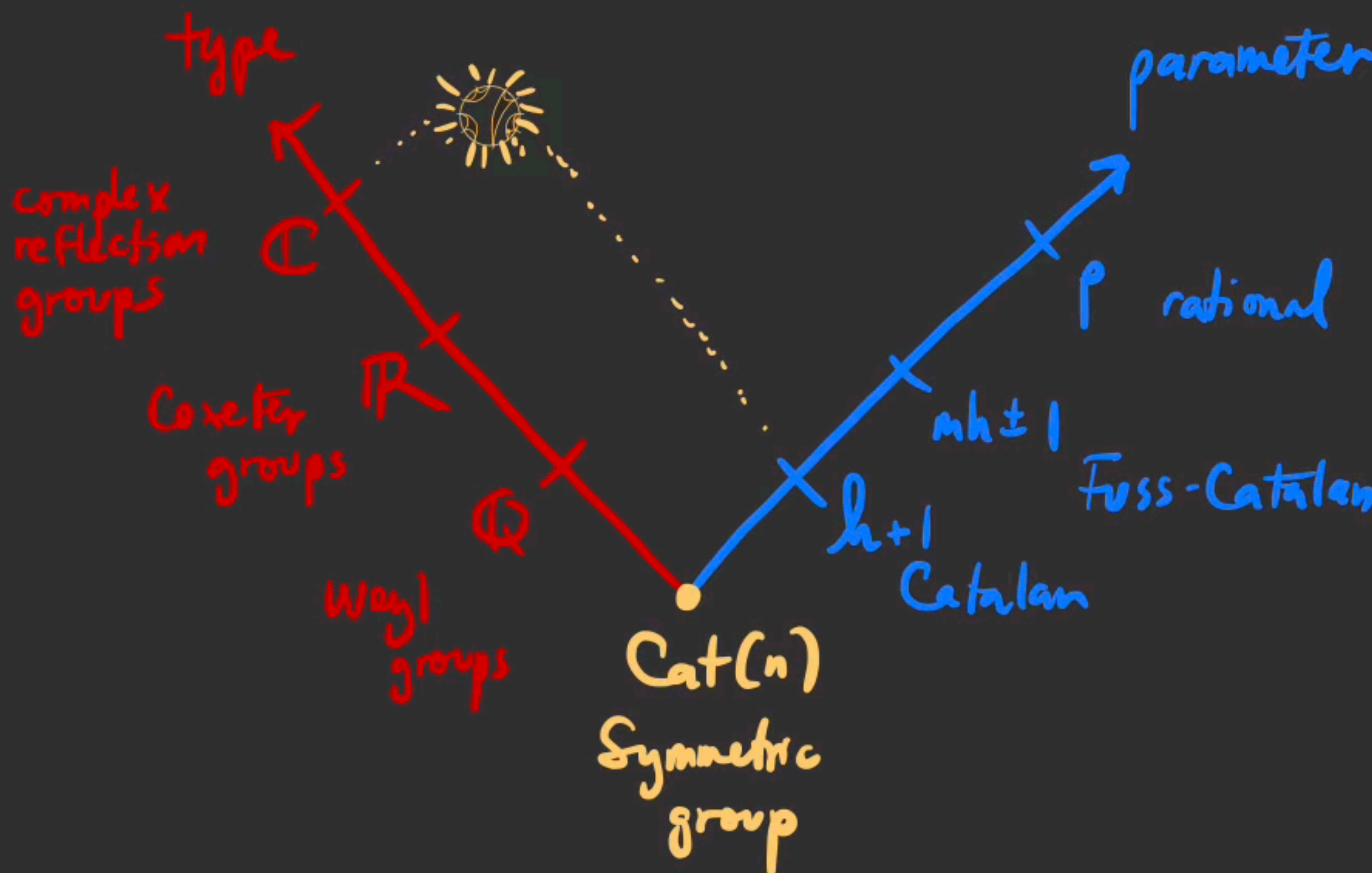
(i) $\mathrm{Hilb}\left(\mathbb{C}[x_1,\dots,x_n]^W\right) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}} = \frac{1}{|W|}\sum_{w\in W} \frac{1}{\det(1-tw)}$

(ii) multiply by $(1-t)^n$: $\prod_{i=1}^{n} \frac{1}{[d_i]} = \frac{1}{|W|}\left(1 + (1-t)*\right)$

identity

(iii) set $t \to 1$

II. Cat(W)


type
complex reflection groups
ℂ
ℝ
Coxeter groups
ℚ
Weyl groups
Cat(n)
Symmetric group
parameter
p
rational
mh ± 1
Fuss-Catalan
h+1
Catalan

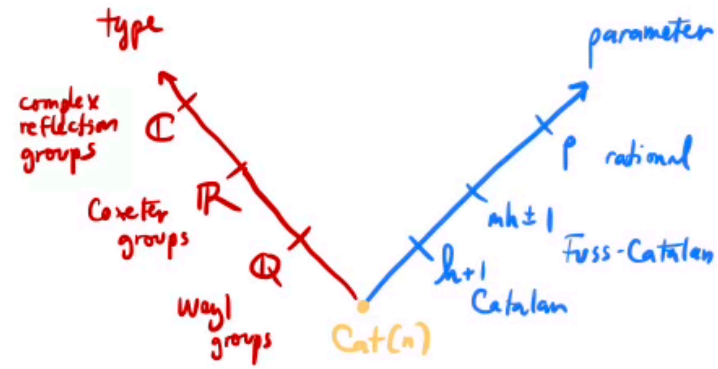
DEF The **Coxeter-Catalan numbers** are the integers

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h+1+e_i}{d_i}.$$

EX $\mathrm{Cat}(n) = \mathrm{Cat}(\mathfrak{S}_n) = \prod_{i=1}^{n-1} \frac{(n+1)+i}{i+1}$

RECALL

"**THM**" (Folklore) Just about every combinatorial object is Catalan.



THM (Reading, Shi/Cellini-Papi)

Only TWO Coxeter-Catalan families:

NC NONCROSSING	NN NONNESTING
	

Cambrian recurrence
Coxeter/well-generated
Dependent on Coxeter element
Hard to change parameter

- noncrossing partitions
- clusters
- sortable elements

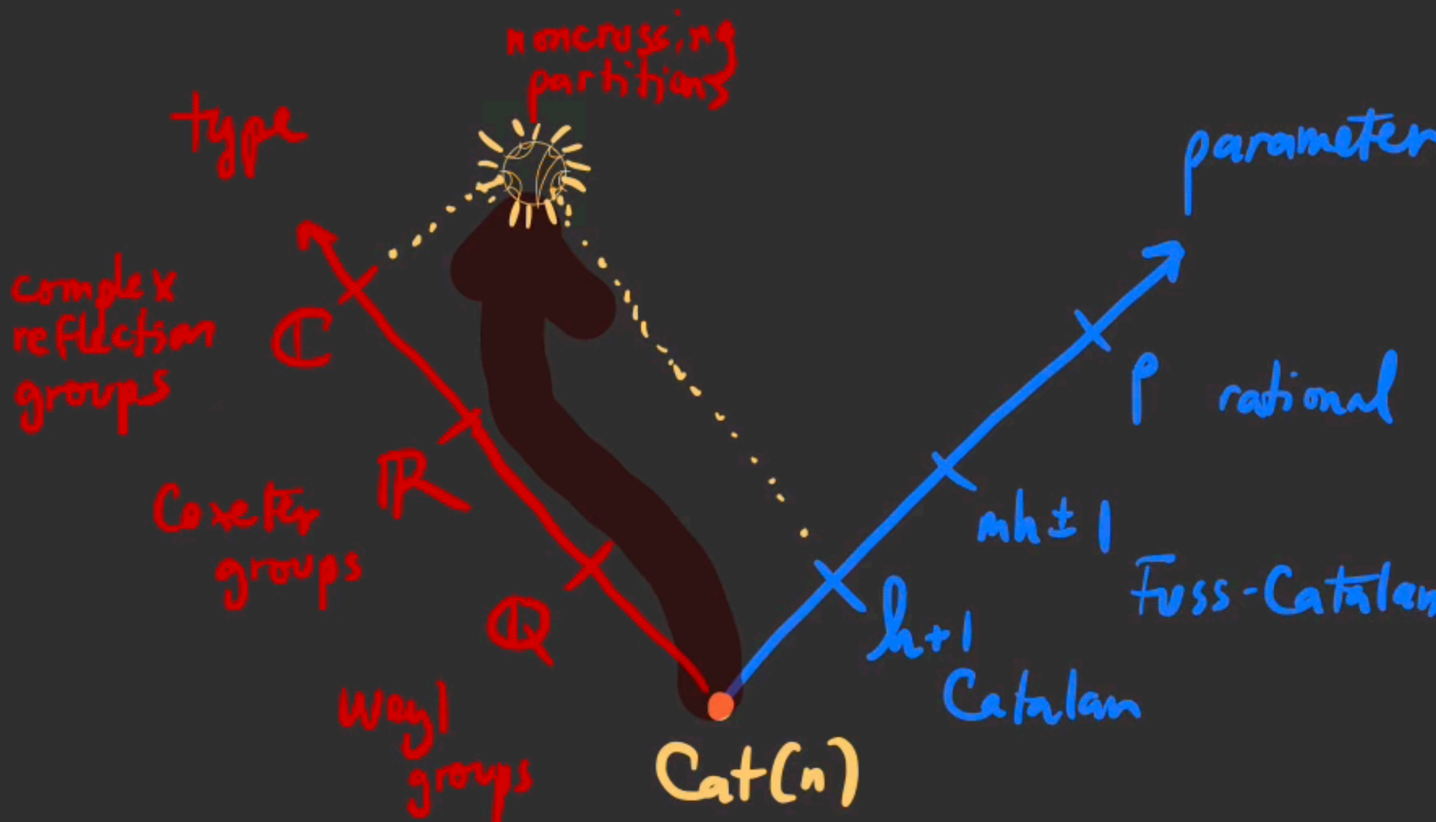
- nonnesting partitions
- dominant Shi regions
- coroots in $(h+1)A_0$

Uniform enumeration
Weyl
Easy to change parameter

REF Reiner. Noncrossing partitions for classical reflection groups.
Reading. Clusters, Coxeter-sortable elements and noncrossing partitions
Bessis. The dual braid monoid

REF Shi. Sign types corresponding to an affine Weyl group
Reiner. Noncrossing partitions for classical reflection groups.
Haiman. Conjectures on the Quotient Ring by Diagonal Harmonics
Cellini, Papi. Ad-nilpotent ideals of a Borel subalgebra II.

III. NONCROSSING PARTITIONS



$\mathbb{R}$-TYPE HISTORY OF NONCROSSING PARTITIONS

1971 - Kreweras. Sur les partitions non croisées d'un cycle.

1993 - Montenegro. The fixed point non-crossing partition lattices

1995 - Reiner. Non-crossing partitions for classical reflection groups

1997 - Birman, Ko, Lee. A new approach to the word problem in the braid groups

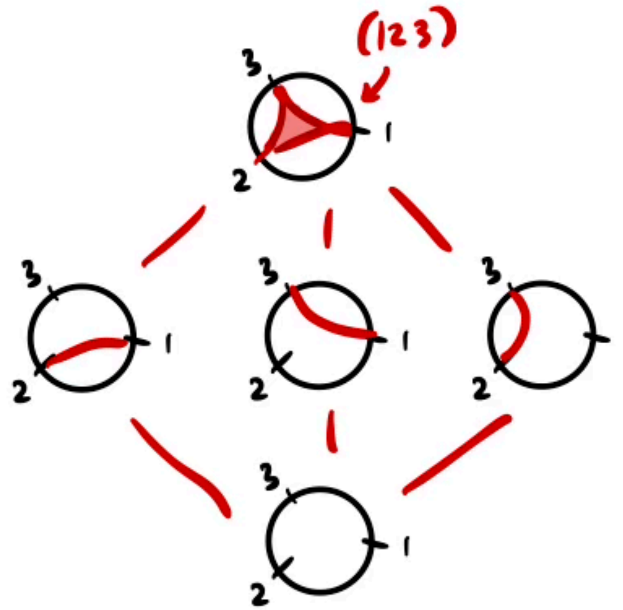
2002 - Brady, Watt. $K(\pi,1)$'s for Artin groups of finite type

2002 - Picantin. Explicit presentations for the dual braid monoids

2003 - Bessis. The dual braid monoid

$\mathbb{R}$- NONCROSSING PARTITIONS

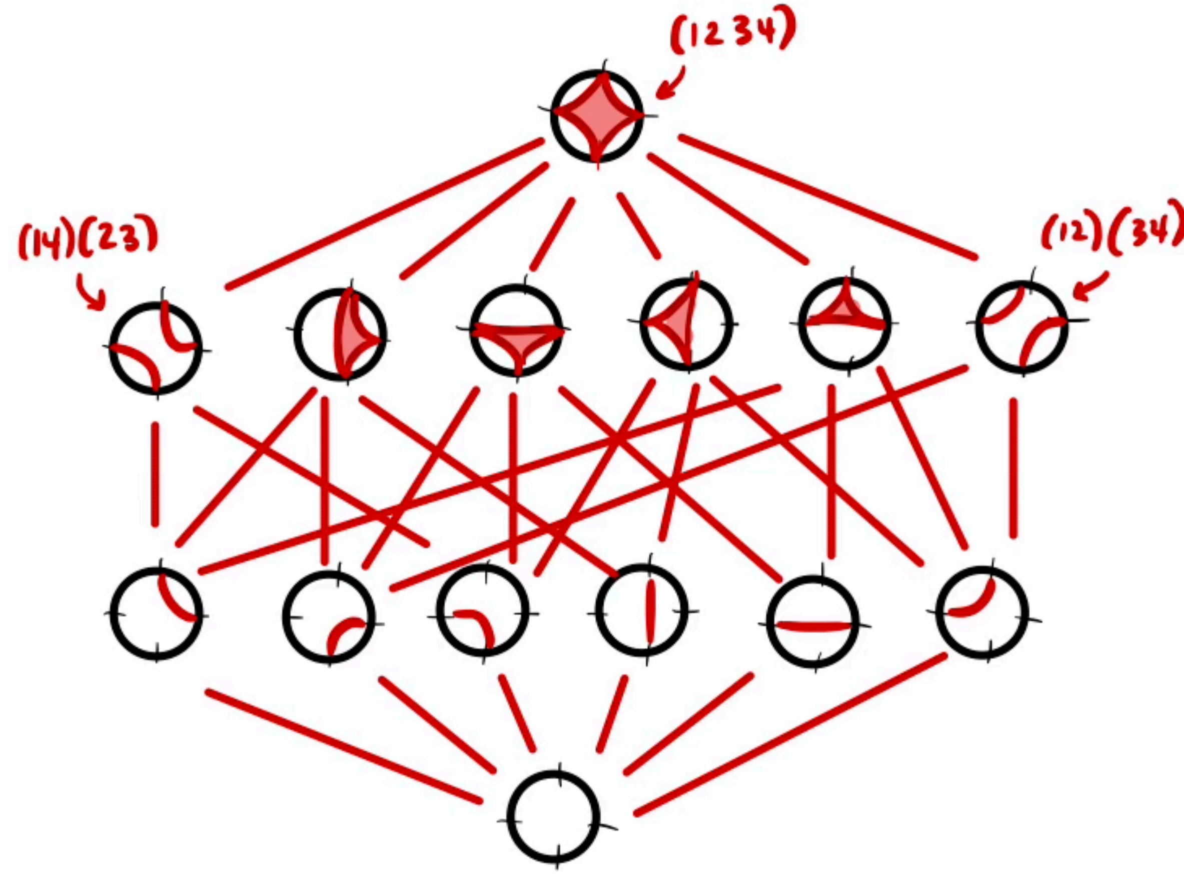
DEF $NC(n)$ = noncrossing (set) partitions ordered by refinement.

$NC(3)$ =  $\subseteq \mathfrak{S}_3$

REF. Kreweras. Sur les partitions non-croisées d'un cycle.

$\mathbb{R}$- NONCROSSING PARTITIONS

<u>DEF</u> $NC(n)$ = noncrossing (set) partitions ordered by refinement.

$NC(4) =$  $\subseteq \mathfrak{S}_4$

$\mathbb{R}$-NONCROSSING PARTITIONS

DEF The **reflections** of W are all conjugates of simple reflections:

$$T = \{ wsw^{-1} \mid s \in S, w \in W \}.$$

DEF A **Coxeter element** c is a product of all simple reflections in some order.

EX In $\mathfrak{S}_n$, $T = \{ (i\ j) \mid 1 \le i < j \le n \}$

$$c = (1 2 \cdots n) \leftarrow \text{the long cycle}$$

FACT The eigenvalues of c in the reflection representation are $\{ \xi_h^{e_i} \}_{i=1}^{n}$

REF Bessis. The dual braid monoid

Reiner. Noncrossing partitions for classical reflection groups.

$\mathbb{R}$-NONCROSSING PARTITIONS

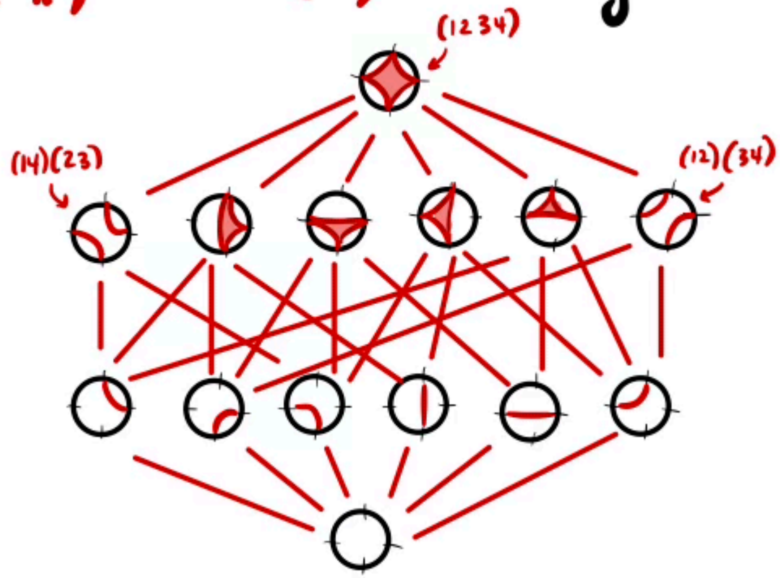
DEF The noncrossing partition lattice is the interval

$NC_c(W) = [e, c]_T$ in the oriented Cayley graph of (W, T).

$\uparrow$ Coxeter element

called absolute order, denoted $\leq_T$

EX $NC_{(12\cdots n)}(\mathfrak{S}_n) \cong NC(n)$ via cycles.



REF Bessis. The dual braid monoid

Reiner. Noncrossing partitions for classical reflection groups.

SUBWORDS & CLUSTERS

THM The subwords of $c\,w_0(c)$ that start at e with n stays and end at w_0 are in bijection with $NC_c(W)$.

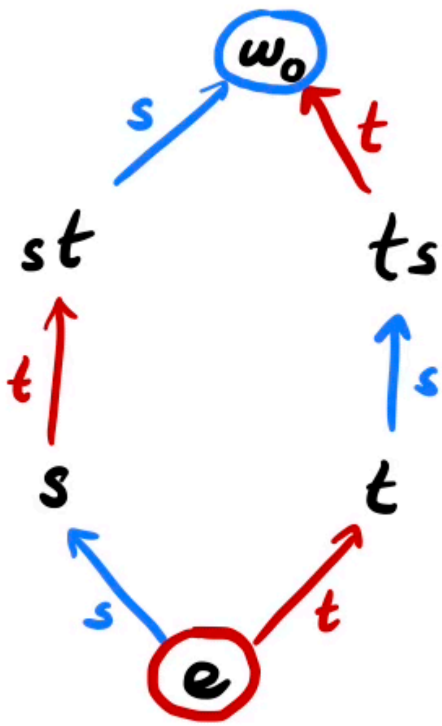
c-sorting word for w_0

long element

(Reading, Ceballos-Labbé-Stump, Pilaud-Stump)

model cluster exchange graph

EX $\mathfrak{S}_3$



$c\,w_0(c) =$

s	t	s	t	s
•	•	s	t	s
s	•	•	t	s
s	t	•	•	s
s	t	s	•	•
•	t	s	t	•

REF. Knutson, Miller. Subword complexes in Coxeter groups

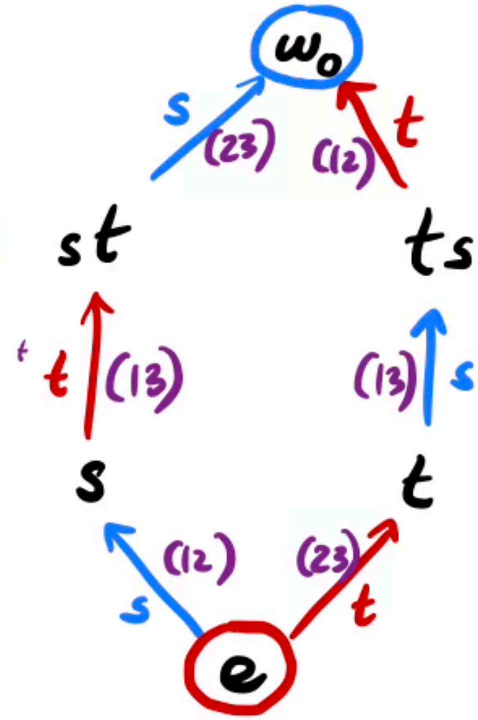
Ceballos, Labbé, Stump. Subword complexes, cluster complexes, and generalized multi-associahedra.

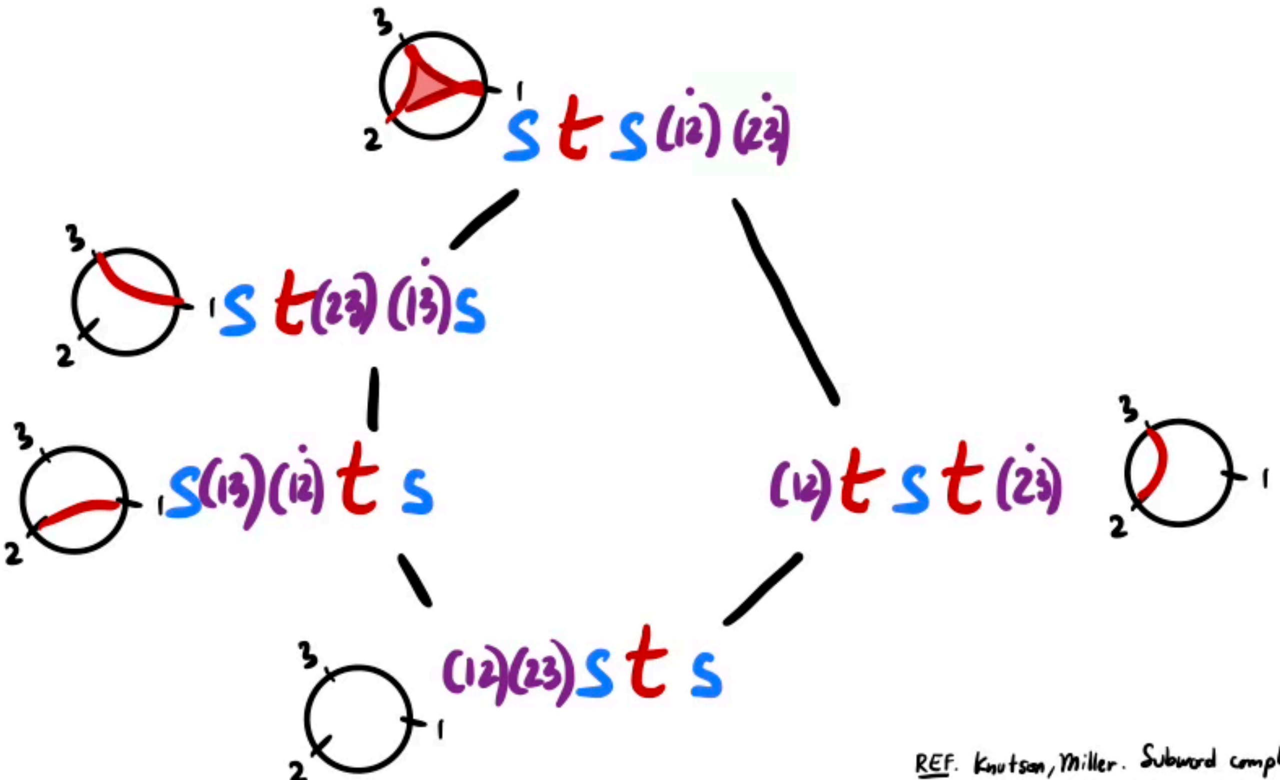
Pilaud, Stump. Brick polytopes of spherical subword complexes and generalized associahedra.

SUBWORDS & CLUSTERS

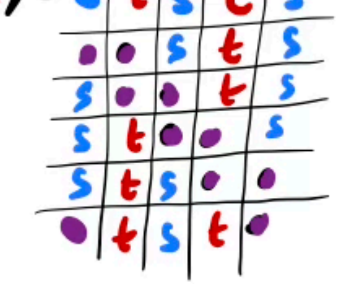
Ex $\mathfrak{S}_3$

BIJECTION: replace stays with colored inversions (root config)





$c\, w_0(c) = s\ t\ s\ t\ s$



REF. Knutson, Miller. Subword complexes in Coxeter groups
Ceballos, Labbé, Stump. Subword complexes, cluster complexes, and generalized multi-associahedra.
Pilaud, Stump. Brick polytopes of spherical subword complexes and generalized associahedra.

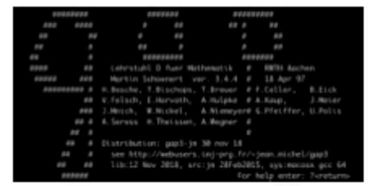
$\mathbb{R}$ - NONCROSSING PARTITIONS

parameter

type

THM
(Conj: Reiner
Proof: Bessis)

$$|NC_c(W)| = Cat(W) = \prod_{i=1}^{n} \frac{h+1+e_i}{d_i}.$$



Proof is NOT UNIFORM: combinatorial models + computer checks
(classical types) (exceptional types)

(In particular, the number of clusters in a cluster algebra of finite type was NOT UNIFORMLY proven to be counted by $Cat(W)$.)

Only TWO Coxeter-Catalan objects

NC-object

HOWEVER...

J. Michel recently found a UNIFORM proof for Weyl groups for a related problem (factorizations of c into reflections)

(i) Chapuy-Stump formula

+

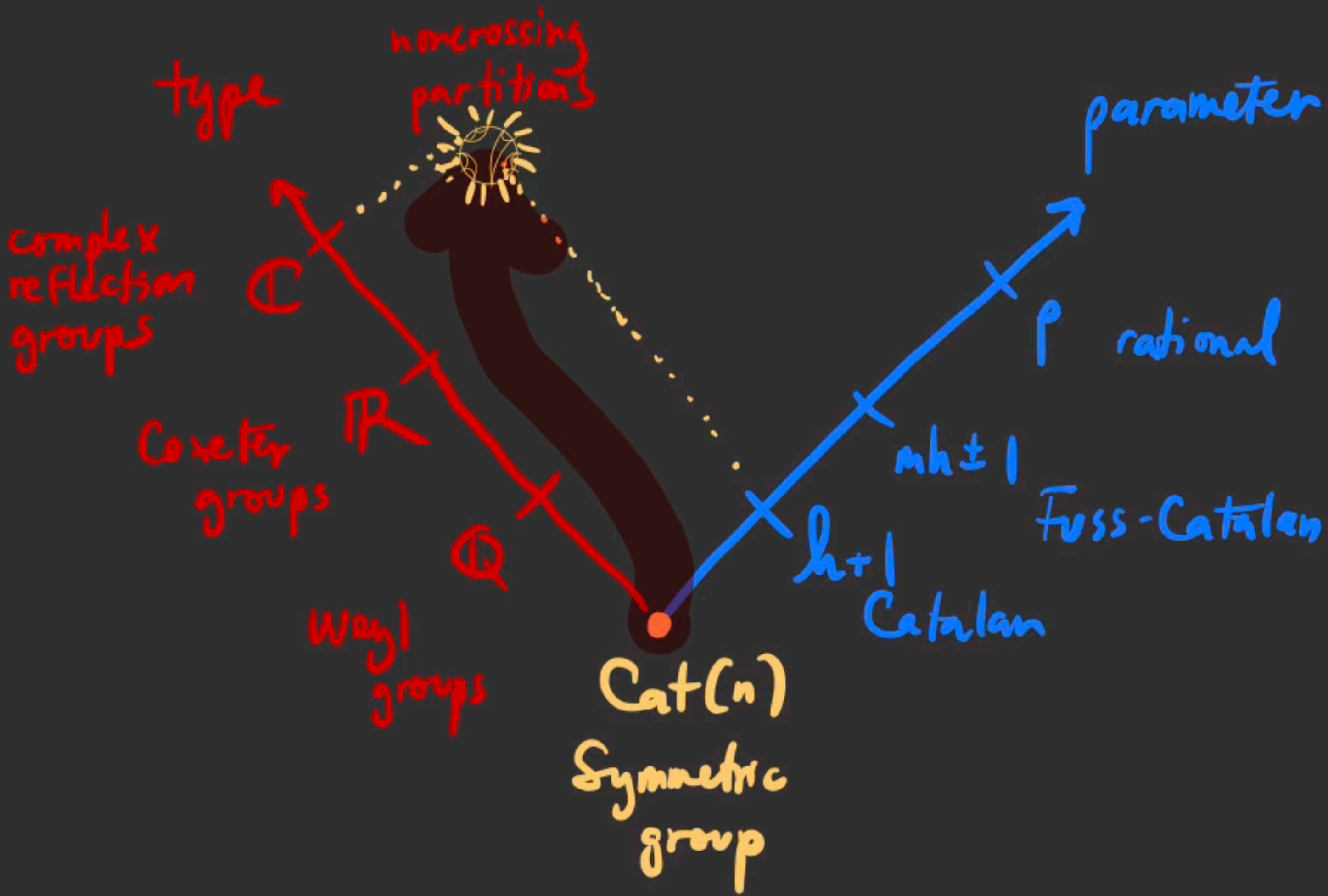
(ii) Frobenius character-theoretic method in Hecke algebra

+

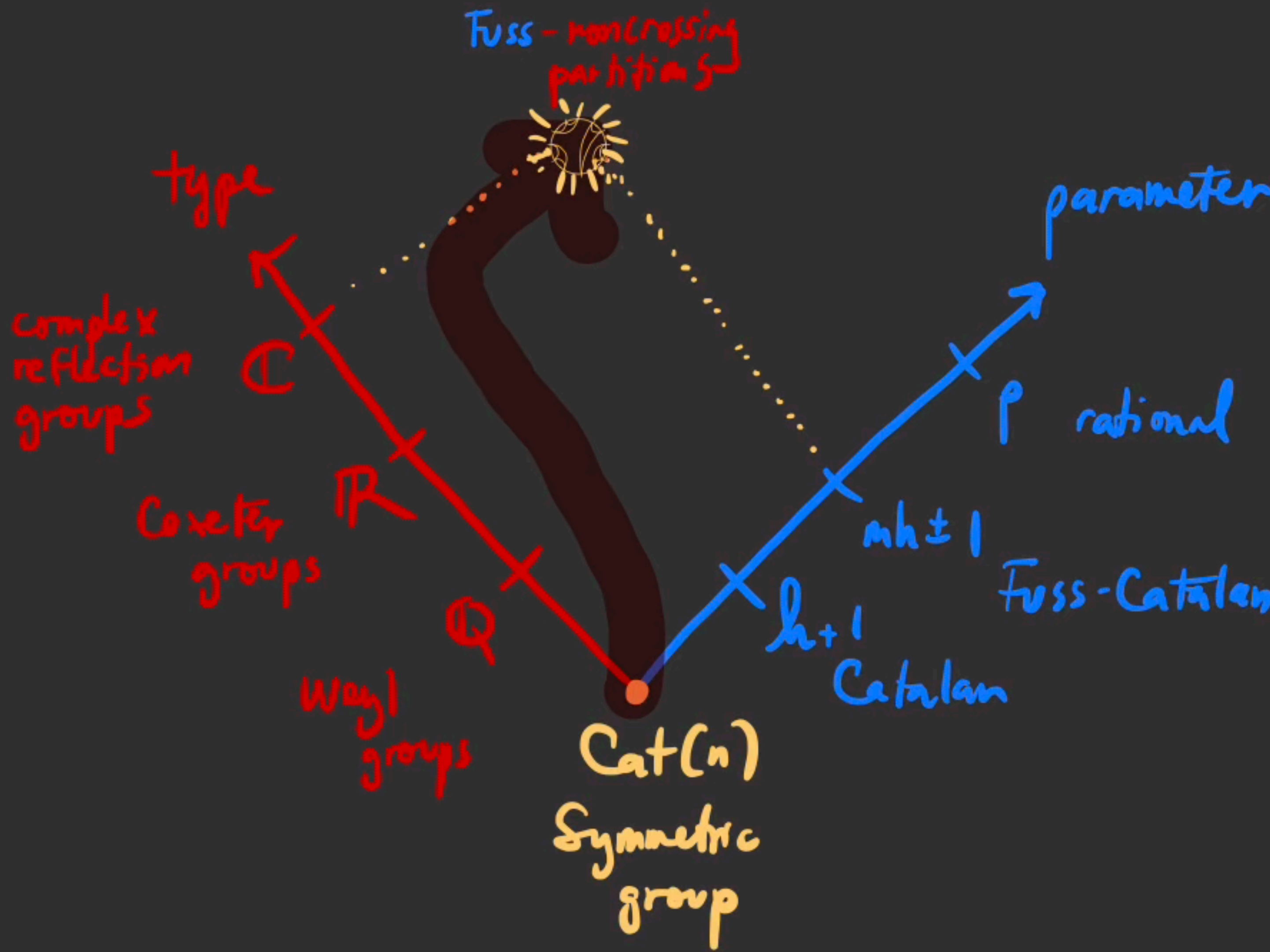
(iii) Deligne-Lusztig theory.

REF Chapuy & Stump. Counting factorizations of Coxeter elements into products of reflections.
Michel. "Case-free" derivation for Weyl groups of the number of reflection factorizations of a Coxeter element!

PROBLEM 1 : uniformly prove $|NC_c(W)| = \prod_{i=1}^{n} \frac{h+d_i}{d_i}$.



IV. WHY THE FUSS?



ℝ-TYPE HISTORY OF NONCROSSING PARTITIONS

1971 - Kreweras. Sur les partitions non croisées d'un cycle.

1980 - Edelman. Chain enumeration and non-crossing partitions.

2007 - Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups

Generalized Noncrossing Partitions and
Combinatorics of Coxeter Groups

Drew Armstrong

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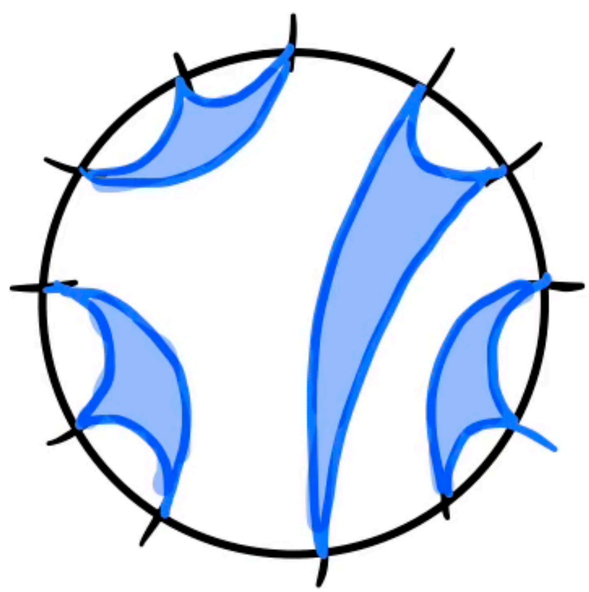
E-mail address: armstron@math.umn.edu

WHY THE FUSS?

DEF $NC_c^m(W) = \{m\text{-multichains in } NC_c(W)\}$.

THM (Armstrong) $|NC_c^m(W)| = \prod_{i=1}^{n} \frac{mh+1+e_i}{d_i}$ (Coxeter-Fuss-Catalan number)

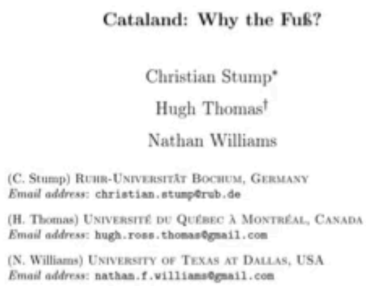
EX $NC_c^m(\mathfrak{S}_n)$ are the m-divisible noncrossing partitions.



REF Edelman, Chain Enumeration and noncrossing partitions
Armstrong, Generalized Noncrossing partitions & Combinatorics of Coxeter groups

WHY THE FUSS?

THM (Stump, Thomas, W.) The subwords of $c\,w_\circ^m(c)$ that start at e with n stays and end at $w_\circ^m = \begin{cases} w_\circ \\ e \end{cases}$ are in bijection with $NC_c^m(W)$.



Cataland: Why the Fuß?

Christian Stump*

Hugh Thomas†

Nathan Williams

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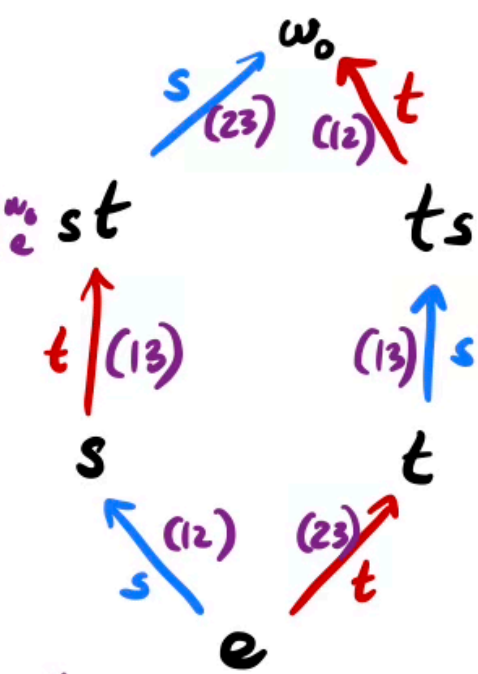
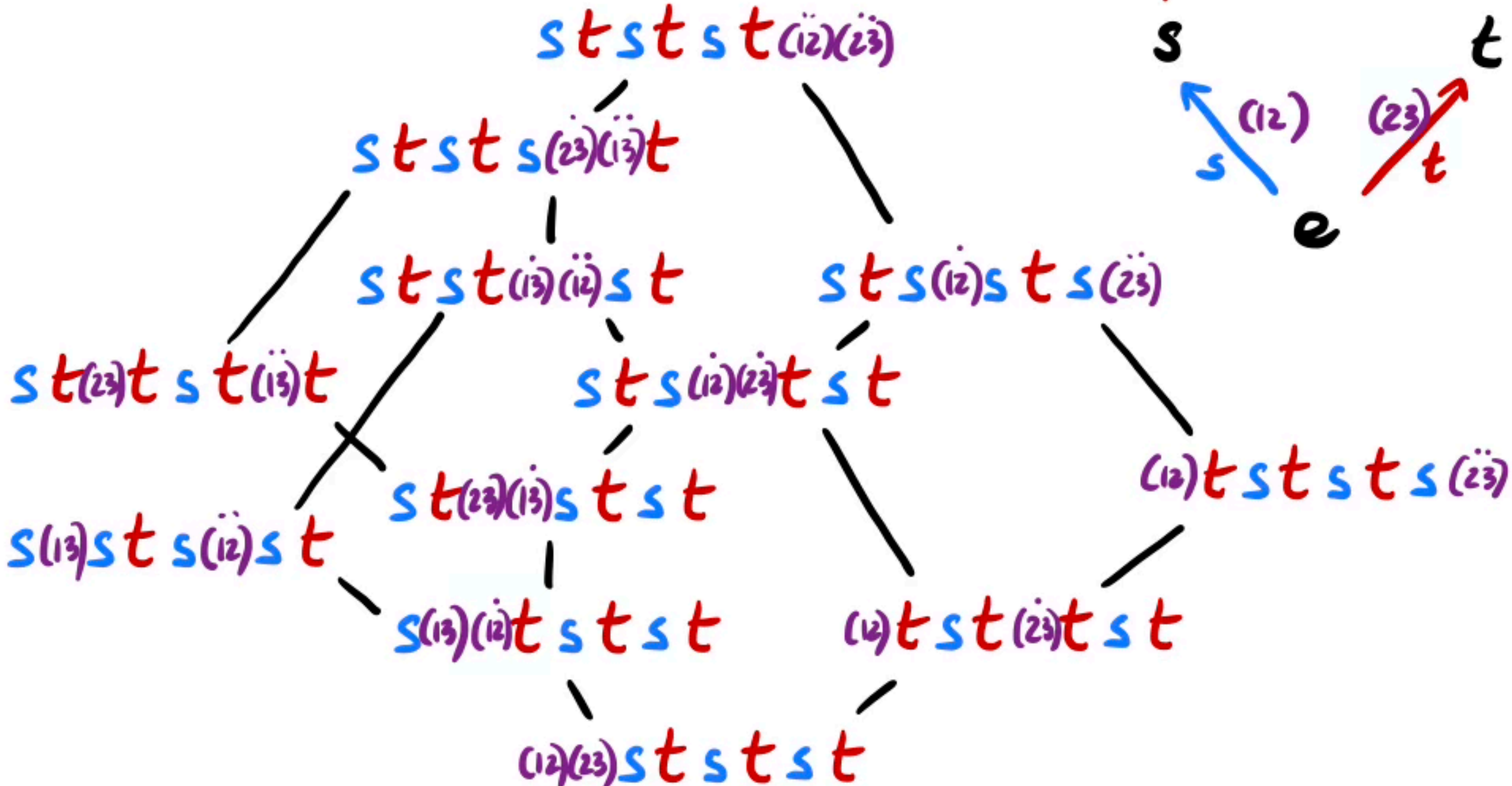




REF Stump, Thomas, Williams. Cataland: why the Fuss?

(Fuss-Dogalan number)

yes, really

Can define $NC_c^{-m}(W)$, counted by $\prod_{i=1}^{n} \frac{mh-1+e_i}{d_i}$.

And then you get stuck. For 10 years.

BUT...

slight lie

BUT... $\displaystyle\prod_{i=1}^{n} \frac{p+e_i}{d_i}$ is ALWAYS an integer for $\gcd(p,h)=1$.

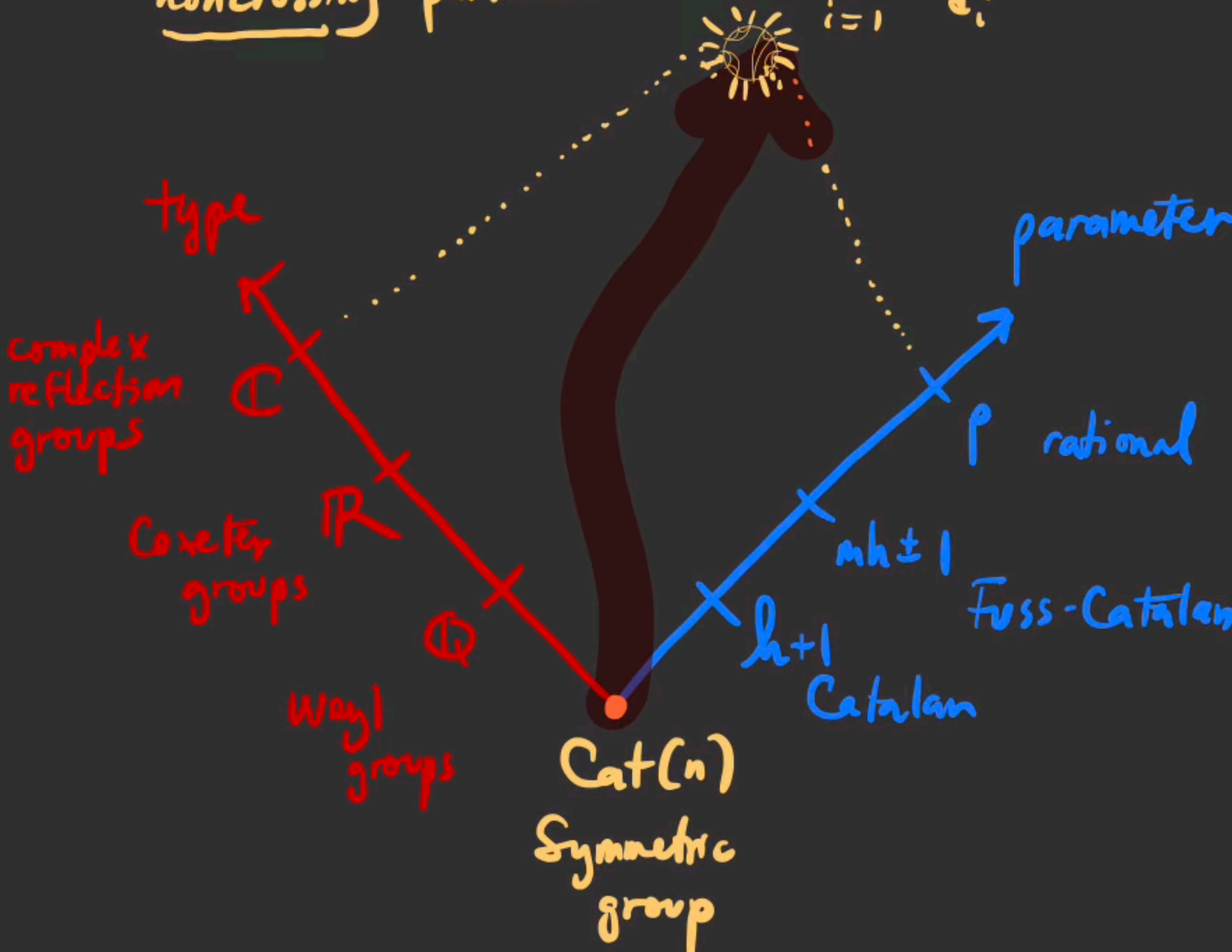
PROBLEM (D. Armstrong, ~2012):

WHAT NC OBJECT IS COUNTED BY $\displaystyle\prod_{i=1}^{n} \frac{p+e_i}{d_i}$???

fractional multichains?
support conditions?
subwords?

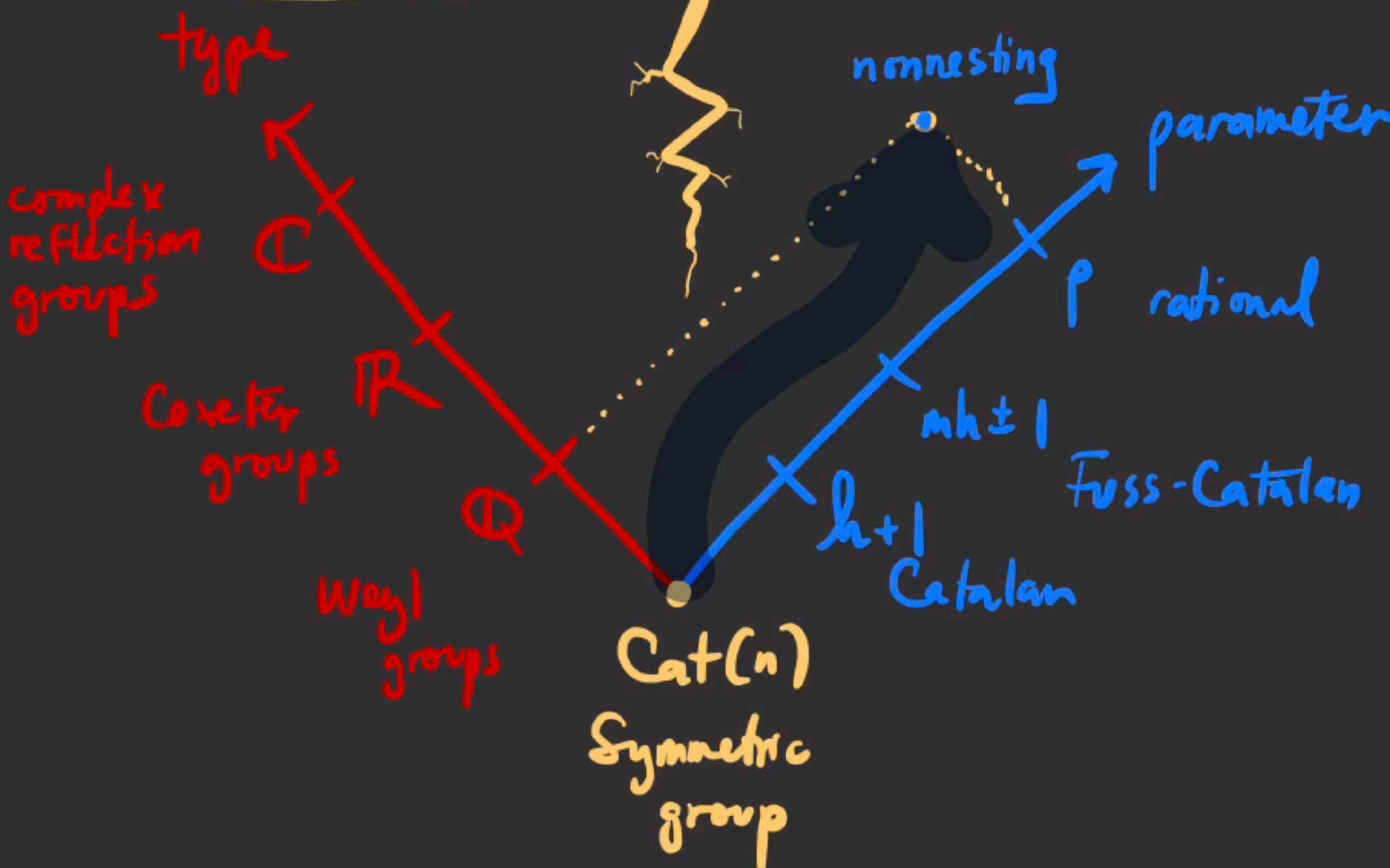
REF Rational Catalan Combinatorics: An Outline from the AIM workshop, Dec 2012

PROBLEM 2: find rational noncrossing partitions $\longrightarrow \prod_{i=1}^{n} \frac{p+e_i}{d_i}$



THM Only TWO types of families: noncrossing & nonnesting.

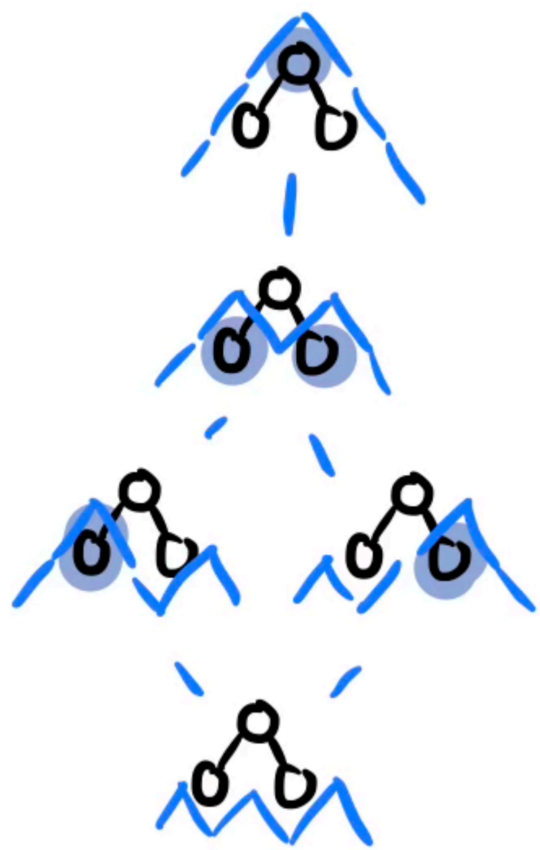
V. NONNESTING PARTITIONS

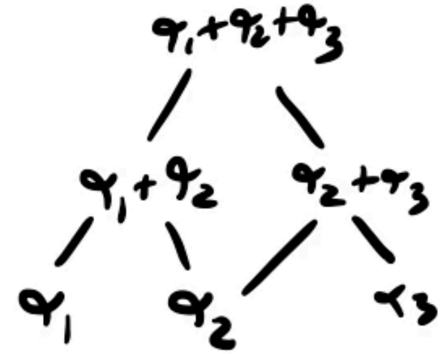


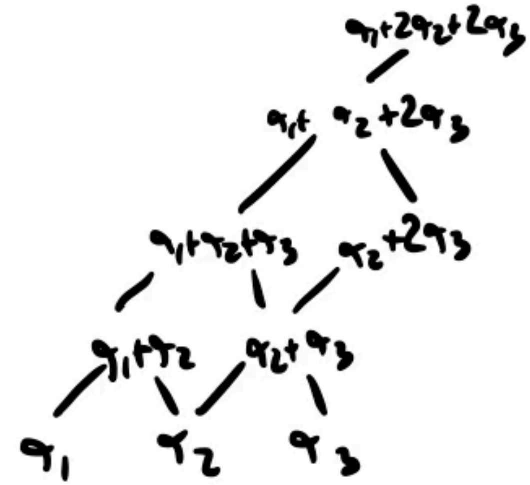
Nonnesting partitions in $\mathfrak{S}_n$

Ex $NN(\mathfrak{S}_n) \cong$ nonnesting (set) partitions (Dyck paths)

$NN(\mathfrak{S}_3) \cong$





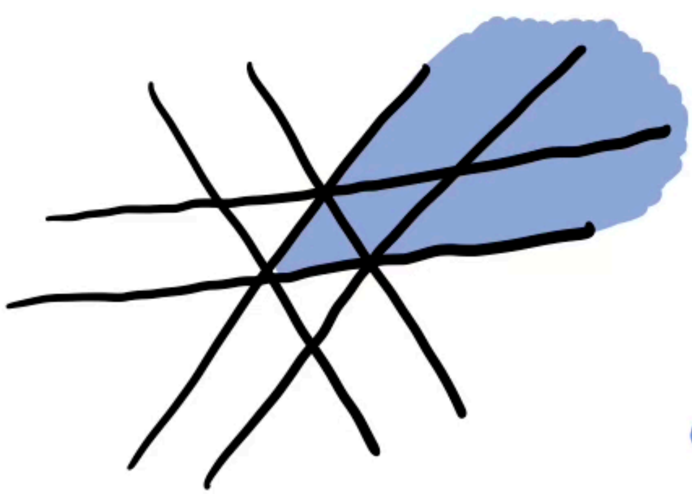


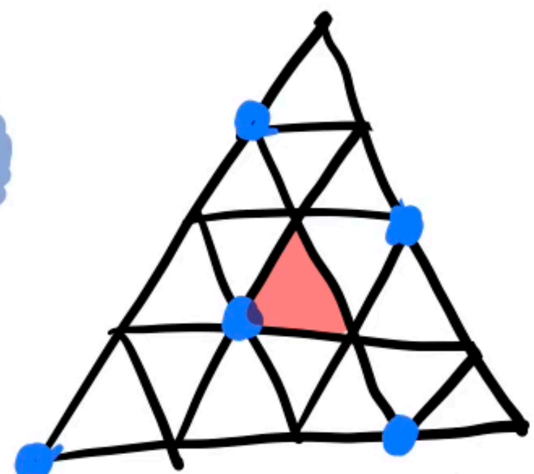
NONNESTING PARTITIONS

<u>DEF</u> (Postnikov) $NN(W) = \{\text{antichains in the positive root poset } \Phi^+\}$.

Weyl group!

<u>THM</u> (Cellini-Papi) $NN(W)$ is in bijection with coroot pts in $(h+1)A_0$.





5 is a Catalan #!

<u>REF</u> Shi. Sign types corresponding to an affine Weyl group
Reiner. Noncrossing partitions for classical reflection groups.
Haiman. Conjectures on the Quotient Ring by Diagonal Harmonics
Cellini, Papi. Ad-Nilpotent ideals of a Borel subalgebra II.

NONNESTING PARTITIONS

DEF For $\gcd(p,h)=1$, $NN^{(p)}(W) = \{\text{coroot pts in } pA_0\}$

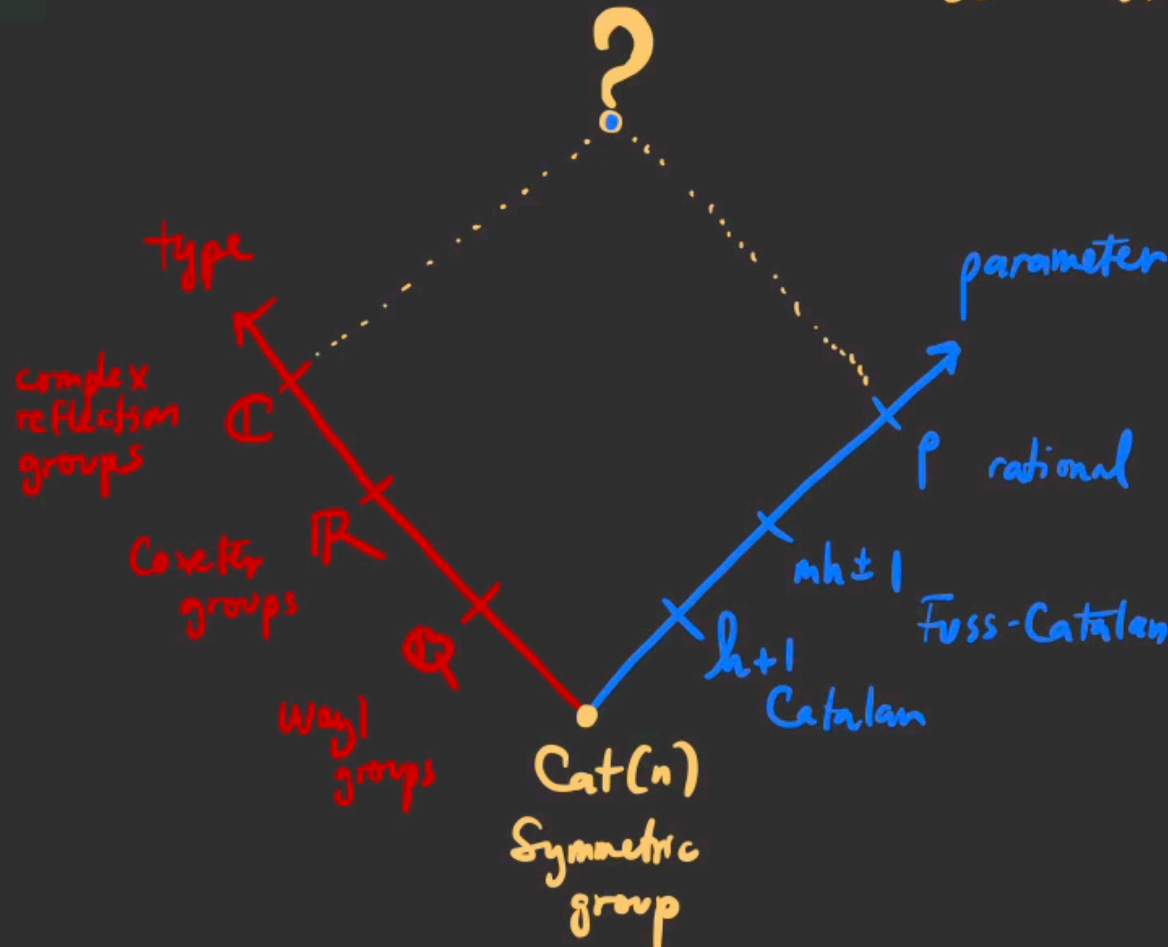
THM (Haiman) (ℚ-UNIFORM) $\left|NN^{(p)}(W)\right| = \prod_{i=1}^{n} \frac{p+e_i}{d_i}$.

REF Haiman. Conjectures on the Quotient Ring by Diagonal Harmonics

OPEN PROBLEM 3 Find nonnesting partitions for complex reflection groups.

OPEN PROBLEM 4 Uniform bijection NC $\leftrightarrow$ NN. I have a candidate using toggles: type A proven by LaCIM!

- Florian Aigner
- Benjamin Dequêne
- Gabriel Frieden
- Alessandro Iraci
- Hugh Thomas



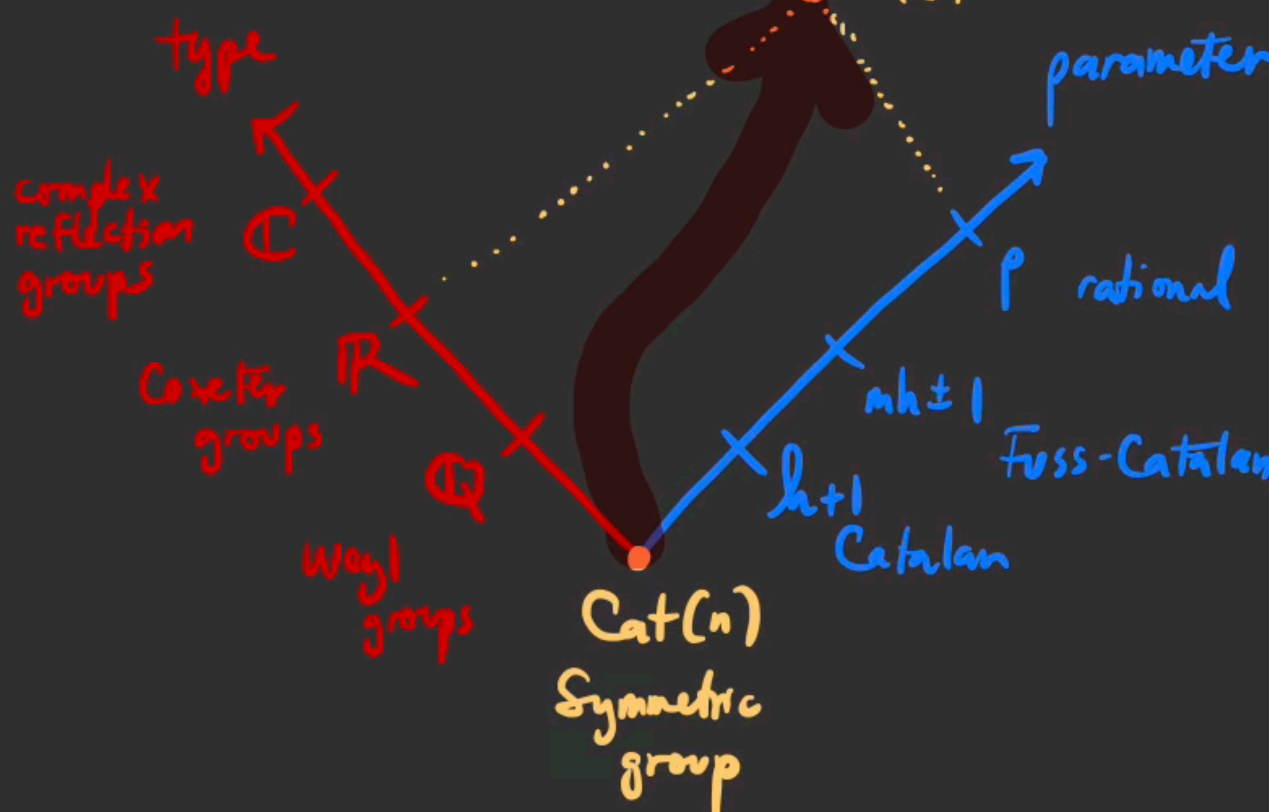
REF. Armstrong, Stump, Thomas. A Uniform Bijection between Nonnesting and Noncrossing partitions.

VII RATIONAL NONCROSSING OBJECTS

Galashin, Lam, Trinh, W.

<u>$\mathbb{R}$-CLOSED</u> : find rational noncrossing partitions $NC_c^{(p)}(W)$.

<u>$\mathbb{Q}$-CLOSED</u> : uniformly prove $\left|NC_c^{(p)}(W)\right| = \prod_{i=1}^{n} \frac{p+e_i}{d_i}$.



RELATIVE POSITION IN THE FLAG VARIETY

Fix G a connected reductive group over $\overline{\mathbb{F}}_q$ with Frobenius F.

For B_1, B_2 Borel subgroups, write $B_1 \xrightarrow{w} B_2$ when

$$(B_1, B_2) \in \left\{ \left({}^{g}B_+ ,\ {}^{g\underset{\text{unique}}{\dot{w}}}B_+ \right) : g \in G \right\}$$

$$\begin{tikzcd}
q^3 & & \underset{w_0}{B_-} & \\
q^2 & st \arrow[ur, "s"] & & ts \arrow[ul, "t"'] \\
q & s \arrow[u, "t"] & & t \arrow[u, "s"'] \\
1 & & B_+ \arrow[ul, "s"] \arrow[ur, "t"'] &
\end{tikzcd}$$

CATALAN VARIETIES (WHAT THE HECKE?)

Write $c = s_1 s_2 \cdots s_n$ for a Coxeter element.

DEF $NCV_c^{(p)}(w) = \Big\{ B_+ = \underbrace{B_0 \xrightarrow{s_1} B_1 \cdots \xrightarrow{s_n} B_n}_{c} \underbrace{\xrightarrow{s_1} B_{n+1}}_{c} \cdots \underbrace{\xrightarrow{s_n} B_{np}}_{c} \xleftarrow{w_0} B_- \Big\}$

Think "c^p"

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

(Q-UNIFORM!)

$$\left| NCV_c^{(p)}(w) \right| = (q-1)^n \prod_{i=1}^{n} \frac{[p+e_i]}{[d_i]}.$$

CATALAN VARIETIES (WHAT THE HECKE?)

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

(Q-UNIFORM!)

$$\left|NCV_c^{(p)}(w)\right| = (q-1)^n \prod_{i=1}^{n} \frac{[p+e_i]}{[d_i]}.$$

PROOF METHOD

(i) Hecke algebra character-theoretic method

\+

Minh-Tâm Trinh (ii) Deligne-Lusztig theory.

similar to J. Michel's proof for the Chapuy-Stump formula

\+

Gordon, Griffeth (iii) Connection to rational Cherednik algebra

REF Gordon, Griffeth. Catalan numbers for complex reflection groups.
Trinh. From the Hecke category to the Unipotent locus

CASE-BY-CASE PROOF VIA LOW-BROW COMPUTATIONS

DEF The Hecke algebra $\mathcal{H}_W$ is the complex associative algebra with basis $\{T_w\}_{w\in W}$ and relations induced by

(i) $T_u T_v = T_{uv}$ if $\ell(u)+\ell(v)=\ell(uv)$ and

(ii) $(T_s+q)(T_s-1)=0$ for $s\in S$.

DEF The trace $\mathrm{tr}:\mathcal{H}_W\to\mathbb{C}[q,q^{-1}]$ is given by

$$\mathrm{tr}(T_w)=\begin{cases}1 & \text{if } w=e\\ 0 & \text{otherwise}\end{cases}$$

CASE-BY-CASE PROOF VIA LOW-BROW COMPUTATIONS

FACT 1 $\left|NCV_c^{(p)}(W)\right| = q^{pn}\, \mathrm{tr}(T_c^{-p})$ (Deodhar recurrence)

Hecke algebra trace $\mathrm{tr}(T_w) = \begin{cases} 1 & \text{if } w = e \\ 0 & \text{otherwise} \end{cases}$

FACT 2 $$\mathrm{tr}(T_c^{-p}) = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{S_\chi(q)} \chi_q(T_c^{-p})$$

Schur elements (formulas + tables exist thank you Götz Pfeiffer!)

FACT 3 $\chi_q(T_c^{-p}) = q^{p h_\chi / h - pn}\, \chi(c)$ for $\gcd(p, h) = 1$

relative Coxeter number

FACT 4 $\chi(c) = 0$ on all but only h many irreps $\{\chi_i\}_{i=1}^{h}$

$$\boxed{\left|NCV_c^{(p)}(W)\right| = \sum_{i=1}^{h} \frac{q^{p h_i / h}}{S_i(q)} \chi_i(c)}$$

REF Geck, Pfeiffer. Characters of Finite Coxeter groups and Iwahori-Hecke alegebras

$$\boxed{\left|NCV_c^{(p)}(w)\right| = \sum_{i=1}^{h} \frac{q^{p h_i/h}}{S_i(q)} \chi_i(c)}$$

Now can evaluate CASE-BY-CASE:

For $\mathfrak{S}_n$, this is a specialization of a computation of V. Jones:

$$\left|NCV_c^{(p)}(\mathfrak{S}_n)\right| = \frac{1}{[n]!} \sum_{i=1}^{n} q^{p(n-i)+\binom{n-i+1}{2}} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} (-1)^{i}$$

$$= (q-1)^{n-1} \prod_{i=1}^{n} \frac{[p+e_i]}{[d_i]} \quad \text{(by } q\text{-binomial theorem)}$$

REF Jones. Hecke algebra representations of braid groups and link polynomials
GAP 3 with CHEVIE

CATALAN VARIETIES (WHAT THE HECKE?)

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

(Q-UNIFORM!)

$$\left|NCV_c^{(p)}(w)\right| = (q-1)^n \prod_{i=1}^{n} \frac{[p+e_i]}{[d_i]}.$$

SHOW ME THE COMBINATORICS!

DEODHAR DECOMPOSITION

Consider the relative position of B_- and B_i for an element of $NCV_c^{(p)}(w)$:

$$
\begin{array}{ccccccccccc}
B_+ & & & & & & & & & & B_- \\
\| & & & & & & & & & & \downarrow w_0 \\
B_0 & \xrightarrow{s_1} & B_1 & \xrightarrow{s_2} & \cdots & \xrightarrow{s_n} & B_n & \xrightarrow{s_1} & B_{n+1} & \xrightarrow{s_2} \cdots \longrightarrow & B_{np} \\
\uparrow u_0 w_0 & & \uparrow u_1 w_0 & & & & \uparrow u_n w_0 & & \uparrow & & \uparrow u_{np} w_0 \\
B_- & & B_- & & & & B_- & & B_- & & B_-
\end{array}
$$

Then $\overset{e}{\overset{\|}{u_0}}, u_1, \ldots, \overset{e}{\overset{\|}{u_{np}}}$ encodes:

(i) a subword of c^p

(ii) that starts and ends at e

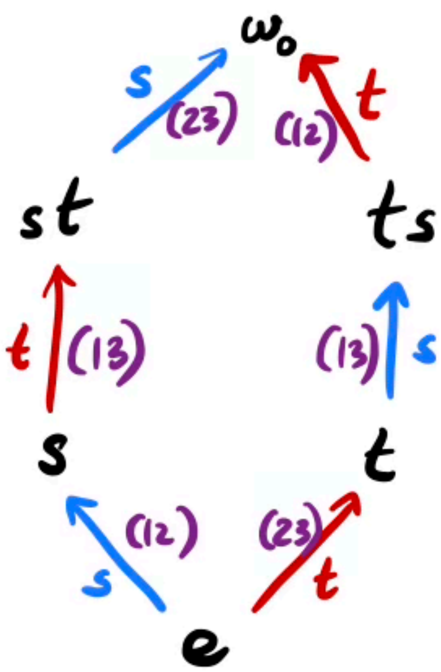
(iii) can stay, $\underbrace{\text{but must go down when possible}}_{\text{no odd colors on stays}}$.

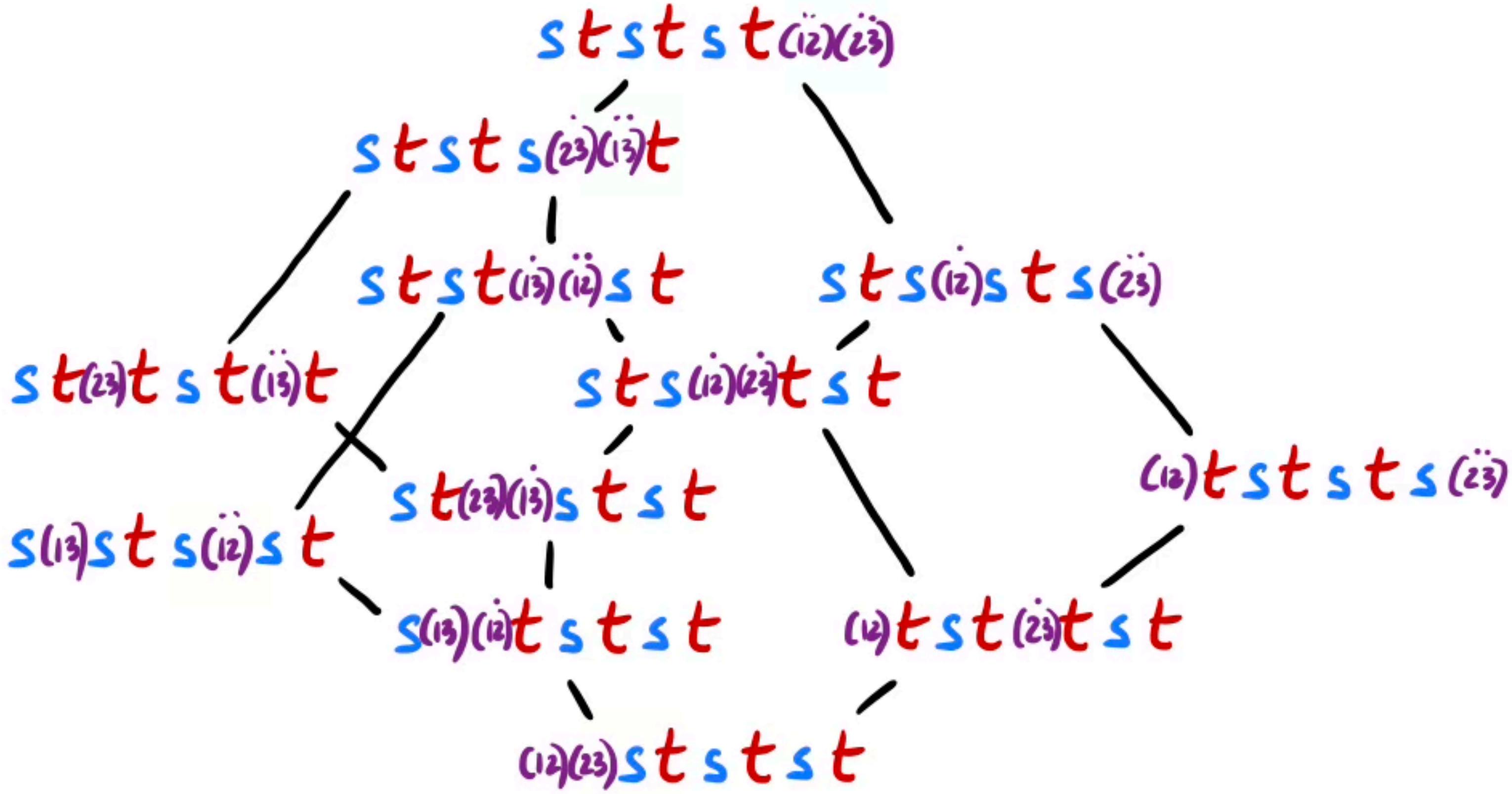
$\Bigg\}$ distinguished subwords $D(c^p, e)$

Ex $W = \mathfrak{S}_3$, $p = 4$

$c = st$

$\underbrace{\text{elements of } D(c^4, e)}_{\text{start \& end at } e\text{, no odd colors on stays}}$ with 2 stays $\equiv$ distinguished

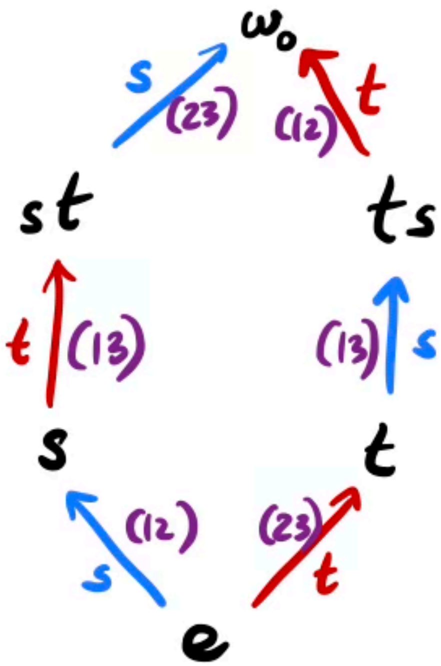


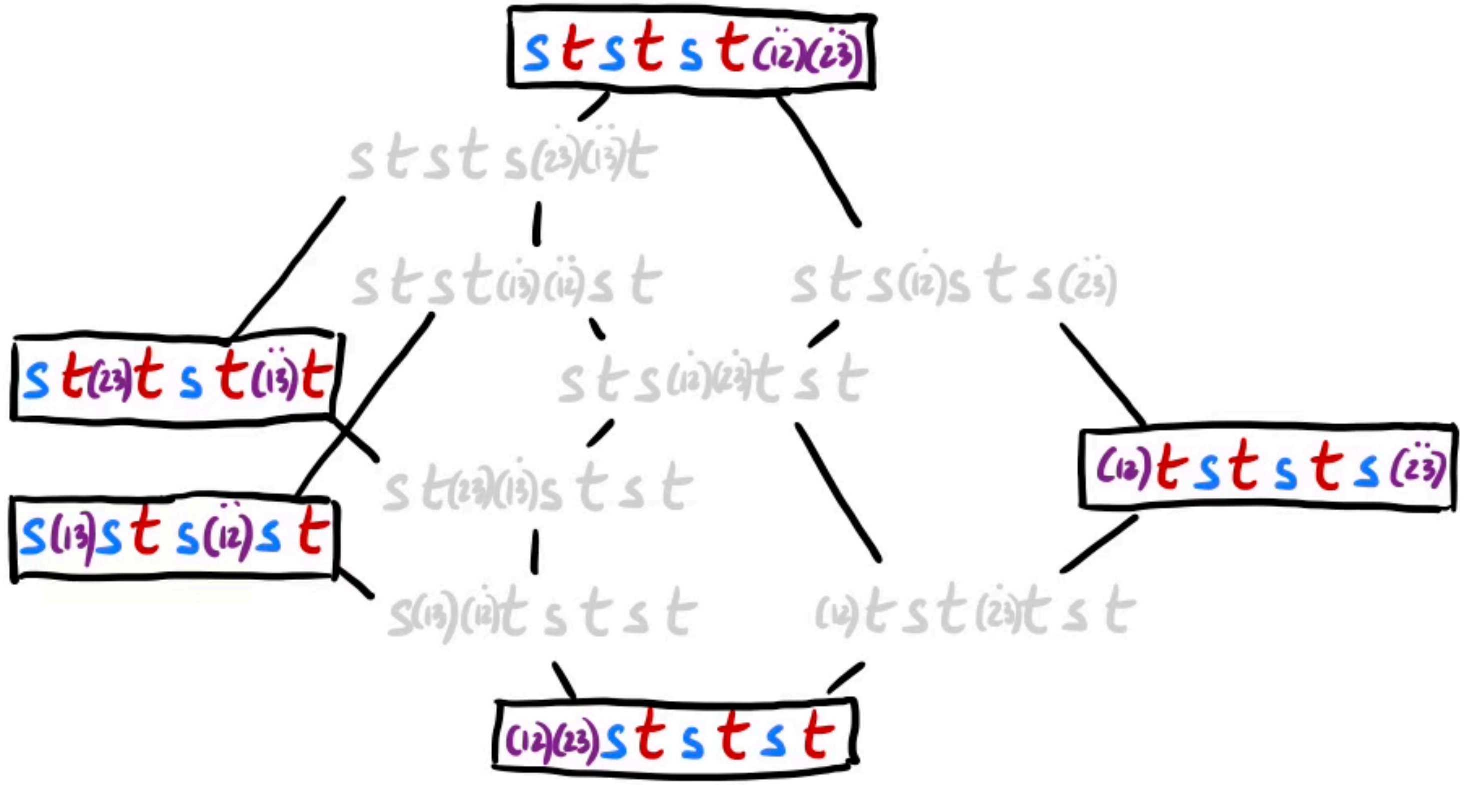


Ex $W = \mathfrak{S}_3$, $p = 4$

$c = st$

elements of $\underbrace{D(c^4, e)}_{\text{start \& end at } e}$ with 2 stays, no odd colors





DEODHAR DECOMPOSITION

DEF $e_u = \#$ stays

$d_u = \#$ descents

THM (Deodhar)

$$\left|NCV_c^{(p)}(w)\right| = \sum_{u \in D(c^p, e)} (q-1)^{e_u} q^{d_u}$$

So

$$\sum_{u \in D(c^p, e)} (q-1)^{e_u} q^{d_u} = (q-1)^n \prod_{i=1}^{n} \frac{[p+e_i]}{[d_i]}$$

want those u for which $e_u = n$ (minimal # stays), then send $q \to 1$.

FROM THE DEODHAR DECOMPOSITION TO COMBINATORICS

DEF $NC_c^{(p)}(w) = \{u \in D(c^p, e) : e_u = n\}$.

$=$ distinguished subwords with exactly n stays

Compute:

$$\sum_{\substack{u \in NC_c^{(p)}(w) \\ (n \text{ stays})}} q^{d_u} + \sum_{\substack{u \in D(c^p, e) \\ e_u > n \\ (\text{more than } n \text{ stays})}} (q-1)^{e_u - n}\, q^{d_u} = \prod_{i=1}^{n} \frac{[p + e_i]}{[d_i]}$$

So at $q = 1$:

$$\left| NC_c^{(p)}(w) \right| = \prod_{i=1}^{n} \frac{p + e_i}{d_i}$$

FROM THE DEODHAR DECOMPOSITION TO COMBINATORICS

I should convince you that $NC_c^{(p)}(w)$ is a noncrossing object.

THM $\underbrace{NC_c^{(mh+1)}(w)}_{\substack{\text{subwords for } e \text{ in } c^{mh+1} = c\,w_0^{2m} \\ \text{with } n \text{ stays and no odd colors}}}$ is in bijection with $\underbrace{NC_c^{m}(W)}_{\substack{\text{subwords for } w_0^m \text{ in } c\,w_0^m(c) \\ \text{with } n \text{ stays.}}}$.

PROOF Halve the colors.

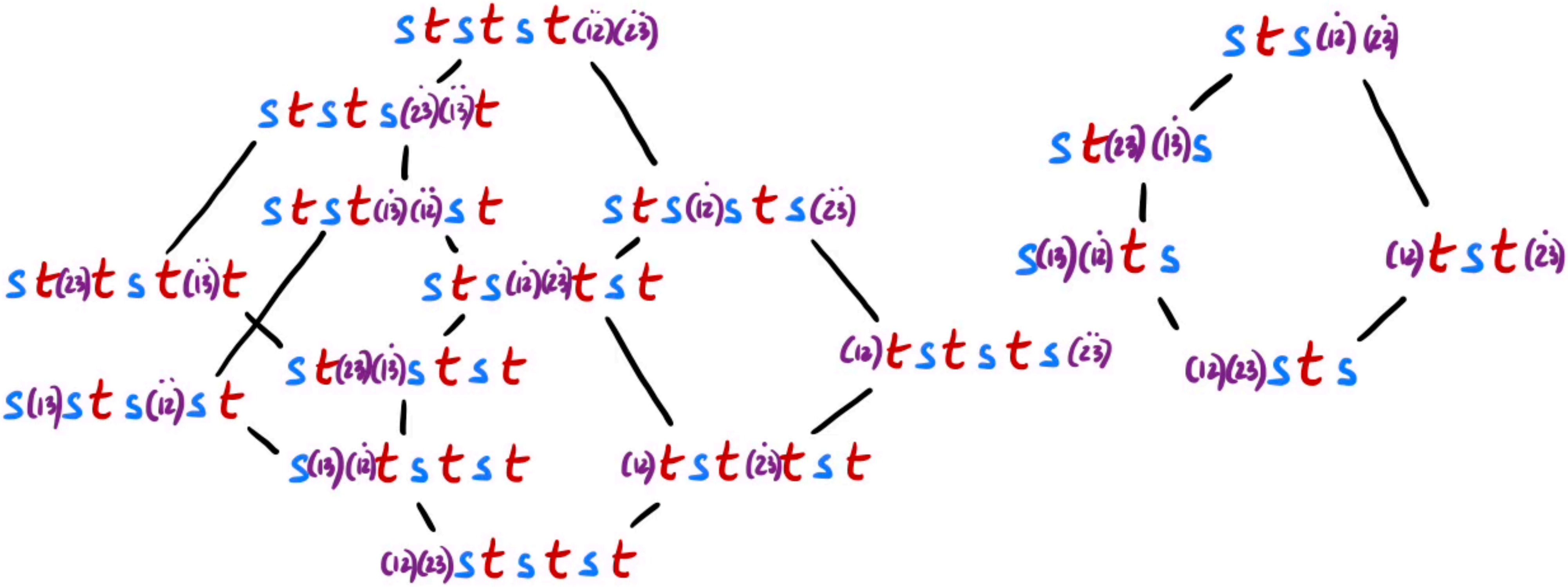
EX $\mathfrak{S}_3$ $m=1$

$NC_c^{(mh+1)}(W)$ $\quad c^{h+1} = c^2$

Subwords for w_0 in $c\,w_0(c)$ with n stays

$NC_c^{m}(W)$

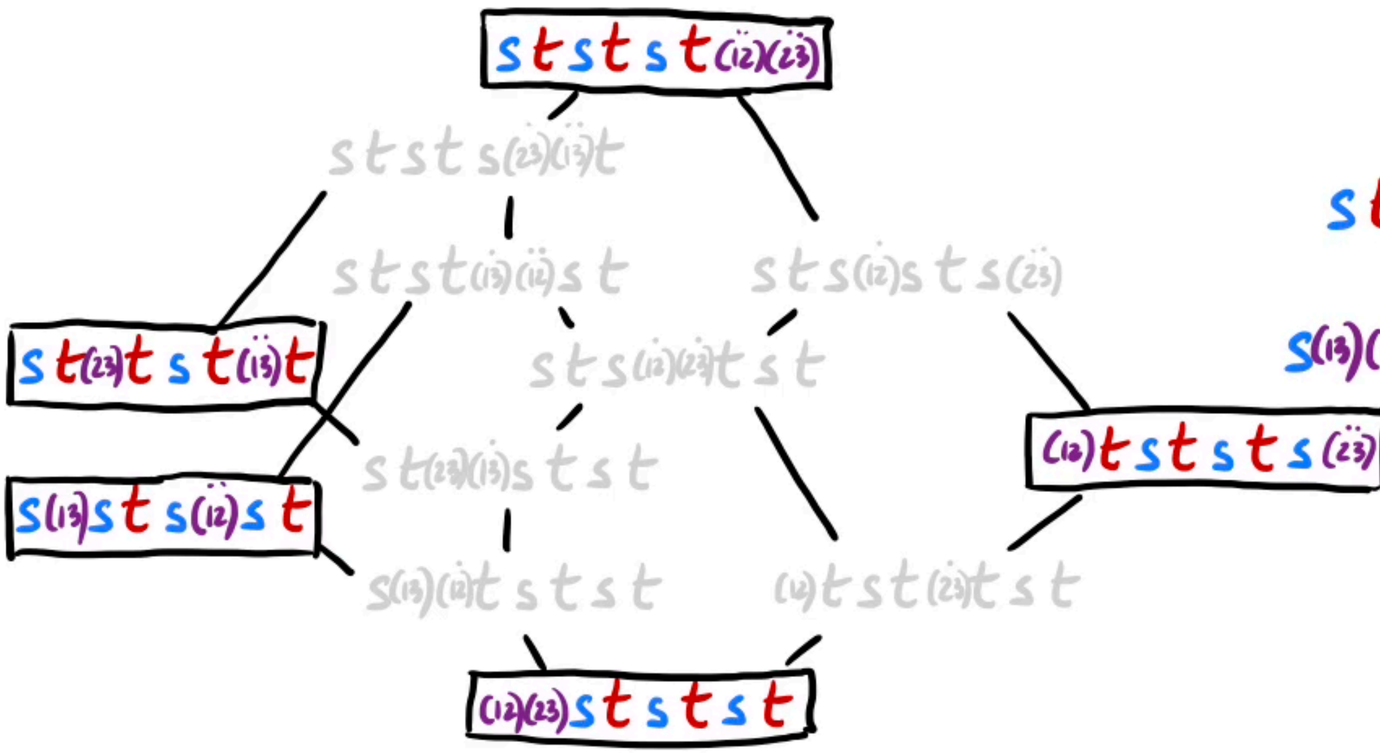
Subwords for w_0 in $c\,w_0(c)$ with n stays



EX $\mathfrak{S}_3$ $m = 1$

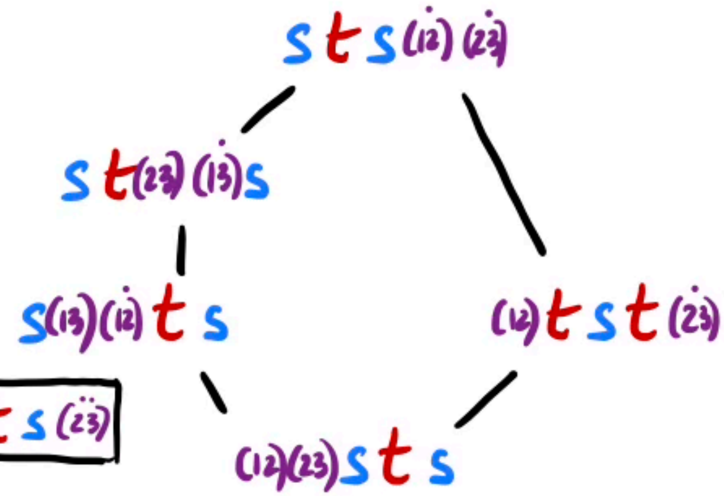
$NC_c^{(mh+1)}(W)$ $\quad c^{h+1} = c^2$

Subwords for w_0 in $c\,w_0(c)$ with n stays and no odd colors



$NC_c^m(W)$

Subwords for w_0 in $c\,w_0(c)$ with n stays



PROBLEM WHAT NC OBJECT IS COUNTED BY $\prod_{i=1}^{n} \frac{p+e_i}{d_i}$?!?

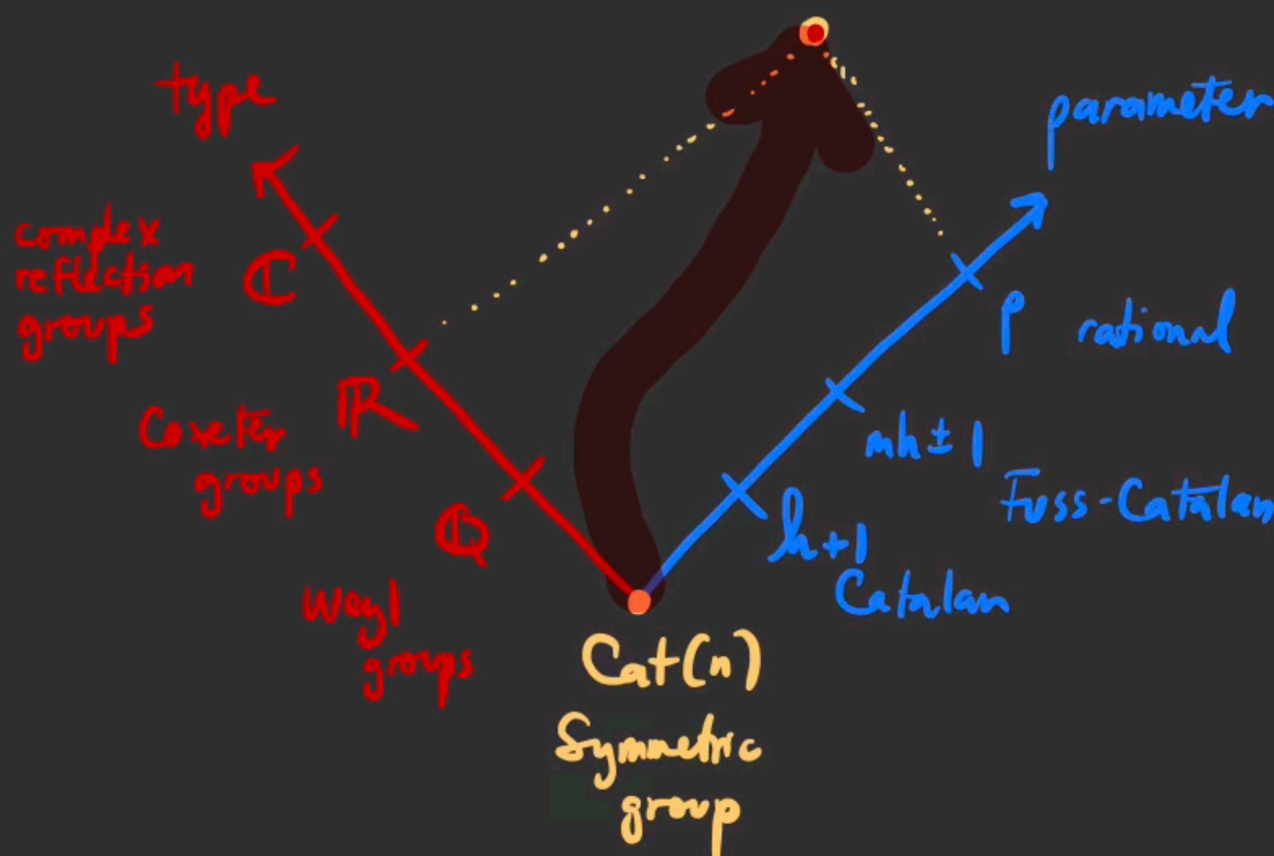
(D. Armstrong, ~ 2012)

THM

(Galashin, Lam, Trinh, W., 2022)

$NC_c^{(p)}(W)$.

OPEN PROBLEM 5: Find combinatorial models for $NC_c^{(p)}(W)$.



WHAT ABOUT PARKING STRUCTURES™?

<u>REF</u> Armstrong, Reiner, Rhoades. Parking Spaces.
Rhoades. Parking Structures: Fuss Analogues.
Edelman, Chain Enumeration and noncrossing partitions

YES!

$$\mathrm{NCPV}_c^{(p)}(W) = \left\{ \begin{array}{cccccccccc} wB_+ = B_0 & \xrightarrow{s_1} & B_1 & \xrightarrow{s_2} & \cdots & \xrightarrow{s_n} & B_n & \xrightarrow{s_1} & B_{n+1} & \xrightarrow{s_2} \cdots \rightarrow B_{np} \\ \uparrow u_0 w_0 & & \uparrow u_1 w_0 & & & & \uparrow u_n w_0 & & \uparrow & \quad \uparrow w w_0 \\ B_- & & B_- & & & & B_- & & B_- & \quad B_- \end{array} \;\middle|\; w \in W \right\}$$

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

(Q-UNIFORM!)

$$\left| \mathrm{NCPV}_c^{(p)}(W) \right| = (q-1)^n \, [p]^n.$$

Again get combinatorial objects as subwords with exactly n skips.

OPEN PROBLEMS

~~Q~~-CLOSED ~~PROBLEM 1~~ 2022: Uniformly prove $|NC_c^{(p)}(W)| = \prod_{i=1}^{n} \frac{p+e_i}{d_i}$.

~~R~~-CLOSED ~~PROBLEM 2~~ 2022: Find rational noncrossing partitions $NC_c^{(p)}(W)$.

OPEN PROBLEM 3: Find nonnesting partitions for complex reflection groups.

OPEN PROBLEM 4: Uniform bijection $NC_c^{(p)}(W) \leftrightarrow NN^{(p)}(W)$ (toggle bijection for $p=h+1$)

OPEN PROBLEM 5: Find noncrossing combinatorial models for $NC_c^{(p)}(W)$, $NCPV_c^{(p)}(W)$ in classical types.

OPEN PROBLEM 6: Follow Galashin's "recipe for success" (compute mixed Hodge cohomology)

OPEN PROBLEMS FOUND WHILE MAKING THESE SLIDES

OPEN PROBLEM 7: Show that the restriction of $\mathrm{Camb}_c^2(W)$ to the Deodar words is isomorphic to $NC_c(W)$.
What happens for $\mathrm{Camb}_c^{2m}(W)$?

BONUS PROBLEMS (R-polynomials)

BONUS PROBLEM 1: In $\widetilde{C}_{2n}$, show $R_{1,t_{2\lambda_n}}(q) = (q-1)^{2n} \sum_{\mu \vdash n} f_\mu^2 \, q^{\mathrm{stat}(\mu)}$

BONUS PROBLEM 2: In C_n, show $R_{1,w_0}(q) = (q-1)^n \sum_{\mu \vdash n} \sum_{T \in SYT(\mu)} q^{\mathrm{stat}(T)}$

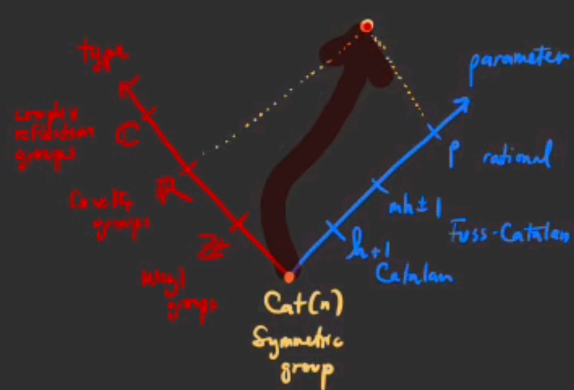
BONUS PROBLEM 3: In $\widetilde{C}_n$, let $\lambda = \sum_{i=1}^{n-1} a_i \alpha_i = \lambda_+ - \lambda_-$ with $a_1 > a_2 > \cdots > a_{n-1} \geq 0$.

Show $R_{t_{\lambda_-}, t_{\lambda_+}}(q) = (q-1)^{n+1} \prod_{i=1}^{n-1} \left(q^{(i+1)a_i - i a_{i+1}} - 1 \right)$.

BONUS PROBLEM 4: Fix $\substack{m > n \\ \gcd(n,m) = 1}$. In $\widetilde{C}_m$ define $w_{n,m} = (s_{m-n+1} \cdots s_{m-1} s_0 s_{m-n} \cdots s_1)^n$

Show $R_{1,w_{n,m}}(q) = (q-1)^{n+m-1} [m]_q^{n-1}$.

THANK YOU !!!



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