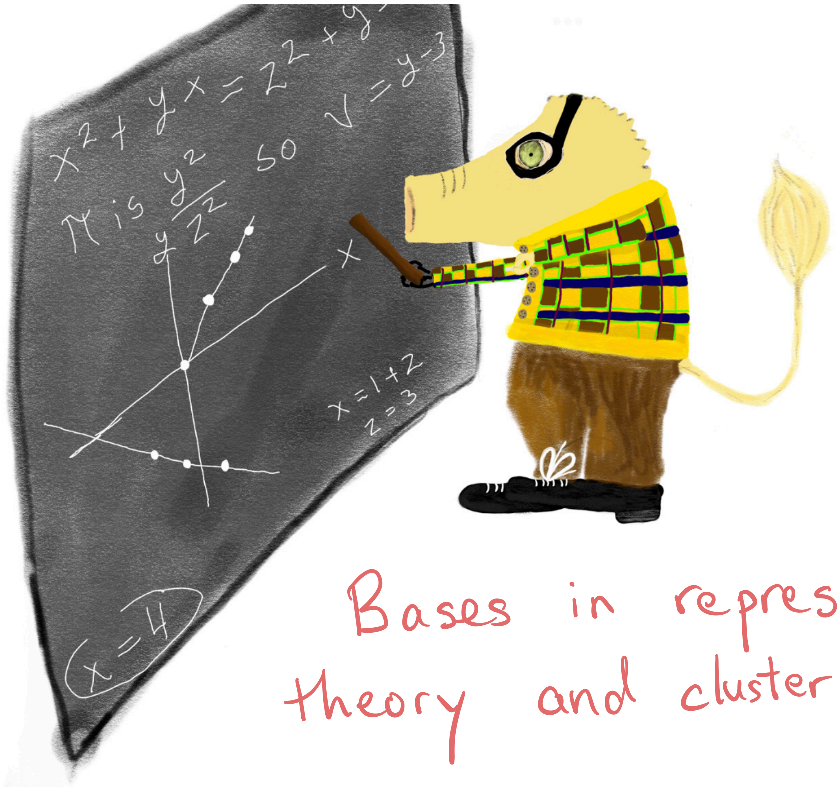


Bases in representation theory and cluster algebras

G semisimple group $/ \mathbb{C}$ eg $G = SL_n \mathbb{C}$

$\Lambda_+ =$ dominant weights $\lambda \in \Lambda_+ \rightsquigarrow V(\lambda)$

eg $\Lambda_+ = \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \}$

$V(\lambda) = \bigoplus_{\mu \in \Lambda} V(\lambda)_\mu$ decomposition under $T \subset G$ $T = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$

$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \Lambda_+} V(\nu)^{\oplus c^\nu_{\lambda\mu}}$

<u>Problem</u> (not open)

Determine combinatorial formulae for weight and tensor product multiplicities.

For $G = SL_n$, these are given by counting certain Young tableaux
In general, solved by Berenstein–Zelevinsky and Littelmann in the 90s

Idea: find bases adapted to these multiplicity spaces

<u>Lemma</u>

$$v \otimes v_\mu + \cdots \longmapsto v$$

$$c^{\nu}_{\lambda\mu} \qquad \mathrm{Hom}(V(\nu), V(\lambda)\otimes V(\mu)) \hookrightarrow V(\lambda)_{\nu-\mu} \qquad\qquad e_i = \begin{bmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 0 \end{bmatrix}$$

$$\phi \longmapsto (1 \otimes v_\mu^*)(\phi(v_\nu))$$

with image $\{ v \in V(\lambda)_{\nu-\mu} : e_i^{\alpha_i^\vee(\mu)+1}(v) = 0 \ \ \forall i \in I \}$

here $\{e_i\}_{i\in I}$ are the Chevalley generators of $\mathfrak{n} \subset \mathfrak{g}$

A basis B for a rep V is called <u>good</u> if

(i) it is a weight basis

(ii) it is compatible with all $\ker e_i^k \subset V$

A good basis for $V(\lambda)$ restricts to a basis of each tensor product multiplicity space $\mathrm{Hom}(V(\nu), V(\lambda)\otimes V(\mu))$

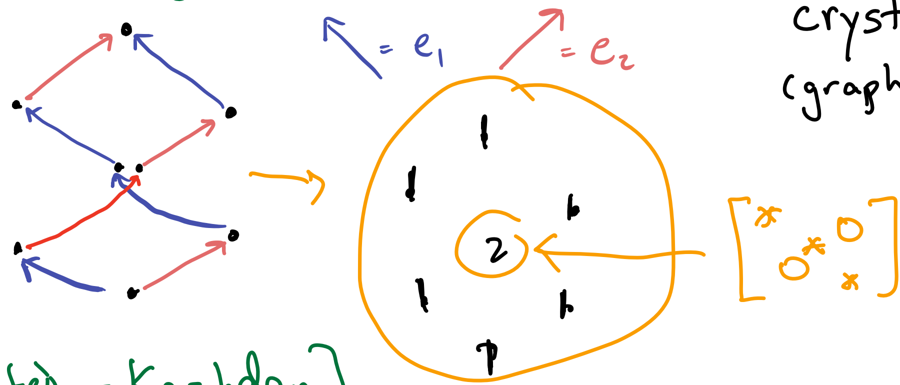
For $v \in V$, $\varepsilon_i(v) = \max\{k : e_i^k(v) \neq 0\}$ $\qquad v \to e_i(v) \to e_i^2(v) \to 0$

A good basis B is called <u>perfect</u> if for all $b \in B$ and $i \in I$, either $e_i(b) = 0$ or $\exists\ \tilde{e}_i(b) \in B$ s.t.

$$e_i(b) = \varepsilon_i(b)\,\tilde{e}_i(b) + \underbrace{v}_{\text{lower order}} \qquad \text{with } \varepsilon_i(v) < \varepsilon_i(b) - 1$$

Given a perfect basis we get a combinatorial structure: set B, $\underline{wt : B \to \Lambda}$, $\tilde{e}_i : B \dashrightarrow B$ $i \in I$

Eg $G = SL_3$ $V = sl_3$



crystal (graph)

Theorem [Berenstein–Kazhdan]
If B and B' are two perfect bases for a representation V, then there is an iso of crystals $B \cong B'$

Problem (not open)
① Find perfect bases for each $V(\lambda)$
② Give combinatorial descriptions of the resulting crystals.

① is hard, no elementary constructions
- dual canonical bases
- dual semicanonical bases

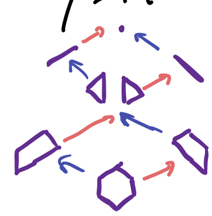
} Lusztig

quiver varieties

- Mirkovic–Vilonen basis

affine Grassmannians

② many combinatorial models for crystals
- Littelmann paths
- MV polytopes



These perfect bases all give the same combinatorics, but

They are different as bases [Kashiwara-Saito, Baumann-Dranowski-K-Knutson, Morton-Ferguson] with the first examples coming in SO_8 and SL_6

Problem

Find the simplest example of a representation with multiple perfect bases.

Instead of studying each representation individually, we can group them together

$N \subset G$

$$V(\lambda) \hookrightarrow \mathbb{C}[N]$$
$$v \longmapsto [g \mapsto v_\lambda^*(gv)]$$
$$v_\lambda \longmapsto 1$$

$\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$

Eg

$G = SL_2$ $\quad V(n) = \mathbb{C}[x,y]_n = Sym^n \mathbb{C}^2$ $\quad V(n) \hookrightarrow \mathbb{C}[y]$, $p(x,y) \mapsto p(1,y)$ $\quad N = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$

$\mathbb{C}[N]$ has left and right actions of $\mathfrak{n}$, e_i, e_i^*

A basis B for $\mathbb{C}[N]$ is called biperfect if it is perfect for the left and right actions.

Theorem [BKK]

$\left(\text{compatible collections of perfect bases for each } V(\lambda) \right) \leftrightarrow \left(\text{biperfect bases for } \mathbb{C}[N] \right)$

- If B, B' are two biperfect bases for $\mathbb{C}[N]$, then $B \cong B'$ as bicrystals.
- For $G = SL_2, SL_3, SL_4$, $\mathbb{C}[N]$ has a unique biperfect basis (but not for SL_6, SO_8)

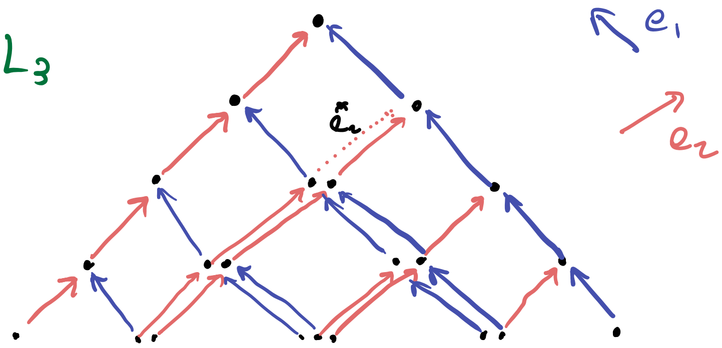
$B(\infty)$ = the bicrystal for $\mathbb{C}[N]$

Knowing $B(\infty)$ solves the tensor product multiplicity problem

$$c^{\nu}_{\lambda\mu} = \#\text{ of } b \in B(\infty) : \varepsilon_i(b) \le \alpha_i^\vee(\mu) \quad \varepsilon_i^*(b) \le \alpha_i^\vee(\lambda) \quad \forall i \in I$$
$$\text{and} \quad wt(b) = \nu - \mu - \lambda$$

Eg

$G = SL_3$



Using MV polytopes / Lusztig data we have

$$B(\infty) \cong G^\vee/B^\vee(\mathbb{Z}_{trop})_{\ge} \qquad \text{[Berenstein-Zelevinsky]}$$
Goncharov-Shen

tropical points of the Langlands dual flag variety, + non-negativity condition

(upper)

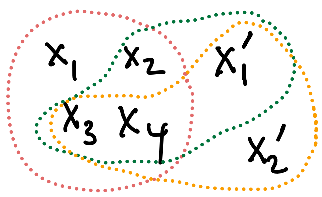
A cluster algebra $\mathcal{A}$ is a commutative algebra along with a collection of seeds $\{x_1, \dots, x_n\}$ s.t. $\mathbb{C}[x_1,\dots,x_n] \subset \mathcal{A} \subset \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$

One seed is obtained from another through mutation.
$$x_1,\dots,x_n \rightsquigarrow x_1', x_2, \dots, x_n$$

Theorem [Berenstein-Fomin-Zelevinsky]
$\mathbb{C}[N]$ has a cluster algebra structure where each reduced word for w_0 gives a cluster (these are only some of the clusters)

long element of the Weyl element

When A is a cluster algebra all cluster monomials are linearly independent.



cluster variables

$x_1^3 x_2^7 x_3 x_4^5 \quad (x_1')^2 x_2' x_3^7$ cluster monomials

Eg $G = SL_3$ two clusters: $\{x, z, \overbrace{xy - z}^{\text{frozen}}\}$
$\{y, z, xy - z\}$

$$N = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix}$$

cluster monomials:
$$\{x^a z^b (xy - z)^c\} \cup \{y^a z^b (xy-z)^c\}$$
this is the unique biperfect basis for $\mathbb{C}[N]$

Theorem [Geiss-Leclerc-Schroer, Kang-Kashiwara-Kim-Oh]
The dual canonical and dual semicanonical bases contain all cluster monomials.

Problem
Does the MV basis contain all cluster monomials?

Baumann-Gaussent-Littelmann proved that for reduced words for w_0 satisfying a certain condition, all cluster monomials in this cluster lie in the MV basis

When the cluster algebra is of finite type, the cluster monomials form a basis.

$\mathbb{C}[N]$ is of finite type only when $G = SL_2, .., SL_5, SO_5$

<u>Problem</u> (not open)
Given a non-finite type cluster algebra A, extend the cluster monomials to a basis of A.

A cluster algebra with cluster $\{x_1, \dots, x_n\}$
$a \in A$ is <u>pointed</u> at $g \in \mathbb{Z}^n$ if

$$a = x^g + \underbrace{\dots}_{\text{lower order with respect to a partial order on Laurent monomials}}$$

A basis for A is called <u>good</u> if it consists of elements which are pointed with respect to each cluster.

Good basis vectors are parametrized g-vectors which transform according to a mutation rule [Fomin-Zelevinsky]

$$\text{Good basis} \longleftrightarrow \bigsqcup_{\text{cluster}} \mathbb{Z}^n/\sim \underset{\substack{\uparrow \\ \text{Fock-Goncharov conjecture}}}{\cong} \text{tropical points of dual cluster variety}$$

Theorem [Qin]
Every good basis contains all cluster monomials

There are three constructions of good bases for cluster algebras

$$\text{Generic bases} \xleftrightarrow{\text{[GLS]}} \underset{\text{bases}}{\overset{\mathbb{C}[N]}{\text{dual semicanonical}}}$$

- come from cluster characters of generic modules in clustercat.

$$\underset{\text{[Qin]}}{\text{Common triangular bases}} \xleftrightarrow{\text{[KKKO]}} \underset{\text{basis}}{\text{dual canonical}}$$

- simple objects in monoidal cat.

$$\underset{\text{[Gross-Hacking-Keel-Kontsevich]}}{\text{Theta basis}} \overset{?}{\longleftrightarrow} \text{MV basis}$$

Problem

Do the theta basis and MV basis coincide?

$\mathbb{C}[N]$

positivity for multiplication

Evidence:

In $\mathbb{C}[N]$ for $G = SL_6, SO_8$ where the bases differ, we have:

$$d = b + v \qquad c = b + 2v$$

b MV basis vector
c dual semicanonical basis vector
d dual canonical basis vector
v vector in all three bases

Leclerc

Same pattern is seen for rank 2 affine type cluster algebras

Problem

Does every biperfect basis contain the cluster monomials?

$$\begin{tikzcd}[column sep=huge]
\text{good basis for } \mathbb{C}[N] \arrow[r, "\text{choose reduced}", "\text{word}"'] \arrow[r, bend right, "\text{g-vector}"] & \mathbb{Z}^m \\
\text{biperfect basis for } \mathbb{C}[N] = B(\infty) \arrow[ur, bend left] \arrow[ur, bend right, "\text{Lusztig data}"'] &
\end{tikzcd}$$

linear transformation

Genz-Koshevoy-Schumann