Combinatorics and Braid Varieties

Nathan Williams

Abstract. We present a framework for finding and proving interesting combinatorial formulas. The combinatorics parametrizes a decomposition of certain braid varieties over finite fields, while the proofs relate the point counts of these varieties to traces in Hecke algebras. We discuss several case-studies and open problems using this framework.

1. Introduction

1.1. Overview. We present a framework for finding and proving interesting combinatorial formulas using braid varieties. This framework is not new [**14**]—we learned about it from Galashin and Lam's work [**18, 19**]—but its application to algebraic combinatorics is new, and has already proven successful in producing new results: recent joint work with Galashin, Lam, and Trinh resolved two decades-long open problems in Coxeter–Catalan combinatorics [**20**].

The framework is based on two objects giving the same q-polynomial:

- the number of points in a braid variety over a finite field $\mathbb{F}_q$ (Definition 2.1); and
- a trace of certain braids in a suitable Hecke algebra (Theorem 3.3).

The first item produces combinatorial objects via the Deodhar decomposition and specializing q to 1 (Section 3.1); the second item allows the use of representation-theoretic techniques for proving enumerative formulas (Section 3.2). At different levels of generality, different techniques become available.

1.2. Summary. In Section 2, we introduce braid varieties, which form the algebraic and geometric background of the framework. In Section 3, we show how to extract combinatorial objects and enumerative formulas from braid varieties.

For finite Weyl and Coxeter groups (Section 4), it is possible to compute everything in a case-by-case manner using an explicit decomposition of the Hecke algebra (Equation (4.1)), and there are many interesting combinatorial and representation-theoretic problems that should be easily resolved using these case-by-case methods. Special classes of elements in finite type (*periodic* elements) have favorable representation-theoretic properties that allow for uniform approaches using Lusztig's exotic Fourier transform, Springer theory, and graded modules of rational Cherednik algebras (Section 4.5). A special case has already led to the solution to two long-standing open problems in Coxeter–Catalan combinatorics in [**20**] (Theorem 4.2).

For affine Weyl groups (Section 5), the main tool available is Opdam's trace formula for translation elements [**35**] (Theorem 5.2). In this setting, the proposed framework recovers some Tesler matrix identities due to Haglund [**24**] (Theorem 5.6)—and we compile several other interesting conjectures of intermediate difficulty (Section 5). Some of these conjectures can be attacked with Opdam's trace formula, while others will require further tools.

For general Weyl groups of Kac–Moody groups (Section 6), we have only rather generic recursive and cluster-theoretic methods. These techniques are not as easy to apply as the specialized trace formulas above, but have the advantage that they provide a tool for studying braid varieties over $\mathbb{C}$, with the aim of computing mixed Hodge decompositions and providing a link to q,t-combinatorics (Section 7).

2. Braid Varieties and the Deodhar Decomposition

We first describe the algebraic part of the framework.

2.1. Braid varieties. Let G be a connected Kac-Moody group split over an algebraically closed field $\mathbb{F}$. Fix a pair of opposed $\mathbb{F}$-split Borel subgroups B_+, B_-. Let $T \coloneqq B_+ \cap B_-$ be a split maximal torus of G, and write $W \coloneqq N_G(T)/T$ for the *Weyl group*. The *flag variety* of G is the collection of Borel subgroups $\mathcal{B} \coloneqq G/B_+$. G acts on the flag variety by conjugation—if $g \in G$ and $B \in \mathcal{B}$, then $g \cdot B \coloneqq gBg^{-1}$.

We write $w \cdot B_+ \coloneqq \dot{w} \cdot B_+$, where $\dot{w} \in G$ is any lift of $w \in W$ to $N_G(T)$. For any two Borels $B_1, B_2 \in \mathcal{B}$, there is a unique w such that $(B_1, B_2) = (g \cdot B_+, gw \cdot B_+)$ for some $g \in G$. In this case, we write $B_1 \xrightarrow{w} B_2$ and say that (B_1, B_2) are in *relative position w*.

Definition 2.1. Let $\mathsf{w} = (s_1, s_2, \ldots, s_m)$ be a word in the simple reflections S of length m, and fix $u \in W$. Define the corresponding *braid variety* by

$$R^{(v)}_{u,\mathsf{w}}(\mathbb{F}) \coloneqq \left\{ (vB_+ = B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_m} B_m \xleftarrow{vuw_\circ} B_-) \mid B_i \in \mathcal{B} \text{ for all } i \right\}.$$

For $v = e$, we write $R_{u,\mathsf{w}}(\mathbb{F}) \coloneqq R^{(v)}_{u,\mathsf{w}}(\mathbb{F})$. When w is the reduced word of an element $w \in W$, we have the special cases that:

- $R_{e,\mathsf{w}}(\mathbb{F})$ is isomorphic to the Schubert cell $B_+\dot{w}B_+/B_+$, and
- $R_{u,\mathsf{w}}(\mathbb{F})$ recovers the open Richardson variety, defined as the intersection of a Schubert and opposite Schubert cells $(B_+\dot{w}B_+/B_+) \cap (B_-\dot{u}B_+/B_+)$.

It is natural to view the word w as an element of the braid group, as there are isomorphisms between $R^{(v)}_{u,\mathsf{w}}(\mathbb{F})$ and $R^{(v)}_{u,\mathsf{w}'}(\mathbb{F})$ when w and w' are related by braid moves.

2.2. Distinguished subwords and the Deodhar decomposition. The variety $R^{(v)}_{u,\mathsf{w}}$ decomposes in understandable pieces using Deodhar's *distinguished subwords*, as we now explain; we closely follow our exposition from [**20**].

Let (W, S) be a Coxeter system with simple reflection S and rank r. The *reflections* T of W are the conjugates of the simple reflections. For $w \in W$, the *length* (resp. *absolute length*) $\ell(w)$ (resp. $\ell_T(w)$) of w is the smallest integer $m \geq 0$ such that w can be expressed as a product of m simple reflections (resp. reflections). For $w \in W$ and $s \in S$, we write $ws < w$ if $\ell(ws) < \ell(w)$ and $ws > w$ if $\ell(ws) > \ell(w)$. The *weak order* on W is the partial order formed by the transitive closure of these relations.

A *subword* of $\mathsf{w} = (s_1, s_2, \dots, s_m)$ is a sequence $\mathsf{u} = (u_1, u_2, \dots, u_m)$ in which $u_i \in \{s_i, e\}$ for all i. For any such sequence, we set $u_{(i)} = u_1 u_2 \cdots u_i \in W$. If $u_{(m)} = u$, then we refer to u as a *u-subword* of w.

Definition 2.2 ([**14, 32**]). For $u \in W$, a u-subword u of w is *v-distinguished* if $vu_{(i)} \leq vu_{(i-1)} s_i$ for all i. We write $\mathcal{D}^{(v)}_{u,\mathsf{w}}$ for the set of v-distinguished u-subwords of w (and set $\mathcal{D}_{u,\mathsf{w}} = \mathcal{D}^{(e)}_{u,\mathsf{w}}$). We define

$$\begin{aligned} \mathrm{e}_{\mathsf{u}} &:= |\{i \in [m] \mid vu_i = e\}|, \\ \mathrm{d}_{\mathsf{u}} &:= |\{i \in [m] \mid vu_{(i)} < vu_{(i-1)}\}|, \text{ and} \\ k &:= \min_{\mathbf{u} \in \mathcal{D}^{(v)}_{u,\mathsf{w}}} \mathrm{e}_{\mathsf{u}}\,. \end{aligned}$$

We write $\mathcal{M}^{(v)}_{u,\mathsf{w}} := \{\mathbf{u} \in \mathcal{D}^{(v)}_{u,\mathsf{w}} \mid \mathrm{e}_{\mathsf{u}} = k\}$ for the set of *maximal v-distinguished u-subwords* (and set $\mathcal{M}_{u,\mathsf{w}} := \mathcal{M}^{(e)}_{u,\mathsf{w}}$).

We now explain some combinatorial facts from [**20**]. When $u = e$, the minimal value k is given by $\ell_T(w)$ [**20**, Proposition 4.8]. For any u, there is a characterization of distinguished subwords using *colored reflection*—that is, a pair $(t, l) \in T \times \mathbb{Z}_{\geq 0}$. Given a subword $\mathsf{u} = (u_1, u_2, \cdots, u_m)$ of a word $\mathsf{w} = (s_1, s_2, \dots, s_m)$ and an index $j \in [m]$, we define the colored reflection

$$t_j(\mathsf{u}) := (s_j^{u_{(j)}}, l_j), \quad \text{where} \quad l_j := \left|\left\{1 \leq i < j \mid s_i^{u_{(i)}} = s_j^{u_{(j)}} \text{ and } u_i \neq e\right\}\right|.$$

We use l dots above a reflection to describe the integer l, and if t is a colored reflection we write $\bar{t}$ for the corresponding uncolored reflection.

Definition 2.3 ([**20**, Definition 4.3]). If $\mathsf{w} = (s_1, s_2, \dots, s_m)$ is a word and $\mathbf{u}$ is a subword of w, then we set

$$\mathsf{inv}(\mathsf{u}) := (t_1(\mathbf{u}), t_2(\mathbf{u}), \dots, t_m(\mathbf{u}))\,.$$

We write $\mathsf{inv}_e(\mathsf{u})$ for the subsequence of $\mathsf{inv}(\mathsf{u})$ obtained by restricting to the indices j for which $u_j = e$.

Proposition 2.4 ([**20**, Propositions 4.4 and 4.7]). *A subword u of a word w is distinguished if and only if each colored reflection in $\mathsf{inv}_e(\mathsf{u})$ has even color. Furthermore, $\prod_{t \in \mathsf{inv}_e(\mathsf{u})} \bar{t} = wu^{-1}$.*

Theorem 2.5 ([**14, 32, 43**]). *Let W be a Weyl group. For a u-subword u of $\mathsf{w} = (s_1, s_2, \dots, s_m)$ we let*

$$R^{(v)}_{\mathbf{u},\mathsf{w}}(\mathbb{F}) := \left\{(vB_+ = B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_m} B_m \xleftarrow{vuw_\circ} B_-) \mid B_- \xrightarrow{vu_{(i)}w_\circ} B_i\right\}.$$

Then we have the Deodhar *decomposition*

$$R^{(v)}_{u,\mathsf{w}}(\mathbb{F}) = \bigsqcup_{\mathsf{u} \in \mathcal{D}^{(v)}_{u,\mathsf{w}}} R^{(v)}_{\mathsf{u},\mathsf{w}}(\mathbb{F}) \quad \textit{with} \quad R^{(v)}_{\mathsf{u},\mathsf{w}}(\mathbb{F}) \simeq (\mathbb{F}^*)^{\mathrm{e}_{\mathsf{u}}} \times \mathbb{F}^{\mathrm{d}_{\mathsf{u}}}. \tag{2.1}$$

Example 2.6. Let $W = \mathfrak{S}_2 = \{e, s\}$. Then $\mathcal{D}_{e,(s,s,s)} = \{(e, e, e), (e, s, s), (s, s, e)\}$ and we find that

$$\sum_{\mathsf{u} \in \mathcal{D}_{e,(s,s,s)}} (q-1)^{\mathrm{e}_{\mathsf{u}}} q^{\mathrm{d}_{\mathsf{u}}} = (q-1)^3 + 2(q-1)q = (q-1)(q^2+1).$$

3. Combinatorial Objects and Formulas

Using the Deodhar decomposition of braid varieties over finite fields, we identify maximal distinguished subwords as the combinatorial objects in our framework. We then describe the number of points of a braid variety over a finite field as a trace in the corresponding Hecke algebra $\mathcal{H}_W$.

3.1. Combinatorial objects. Suppose that $\mathbb{F} = \mathbb{F}_q$ is a finite field with q elements, where q is a prime power. Then we have

$$\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = \sum_{\mathsf{u}\in\mathcal{D}_{u,\mathsf{w}}^{(v)}} (q-1)^{\mathrm{e}_\mathsf{u}} q^{\mathrm{d}_\mathsf{u}}.$$

By definition of k and $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ (see Definition 2.2), we conclude that

$$\lim_{q\to 1} \frac{1}{(q-1)^k}\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = \left|\mathcal{M}_{u,\mathsf{w}}^{(v)}\right|. \tag{3.1}$$

Thus, any technique for computing $\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right|$ gives a formula for the combinatorial set $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ of maximal distinguished subwords—in certain settings, we will even be able to identify $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ with existing combinatorial objects. Such identifications are desirable, but can often be difficult to find.

Problem 3.1. *Suppose* $\frac{1}{(q-1)^k}\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right|$ *has nonnegative coefficients.*

- *Relate the maximal distinguished subwords* $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ *to existing combinatorial objects.*
- *Find a combinatorial statistic* stat *on* $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ *so that*

$$\frac{1}{(q-1)^k}\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = \sum_{\mathbf{u}\in\mathcal{M}_{u,\mathsf{w}}^{(v)}} q^{\mathrm{stat}(\mathbf{u})}.$$

Example 3.2. Continuing Example 2.6, we have $\mathcal{M}_{e,(s,s,s)} = \{(e,s,s),(s,s,e)\}$. Problem 3.1 asks for a way to assign a statistic so that

$$\frac{1}{(q-1)^k}\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = 1+q^2 = q^{\mathrm{stat}(e,s,s)} + q^{\mathrm{stat}(s,s,e)} = \sum_{\mathbf{u}\in\mathcal{M}_{e,(s,s,s)}} q^{\mathrm{stat}(\mathbf{u})}.$$

3.2. Formulas. Every braid w in the simple reflections of W gives rise to a corresponding element T_w of the Hecke algebra, and it turns out that the point count $R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)$ can be expressed in terms of T_w.

The *Hecke algebra* of (W,S) is the $\mathbb{Z}[q^{\pm 1}]$-algebra $\mathcal{H}_W$ freely generated by symbols T_w for $w \in W$, modulo the relations

$$T_w T_s = \begin{cases} qT_{ws} + (q-1)T_w & \text{if } ws < w, \\ T_{ws} & \text{if } ws > w, \end{cases}$$

for all $w \in W$ and $s \in S$. For any word $\mathsf{w} = (s_1, s_2, \ldots, s_m)$, we set $T_\mathsf{w} \coloneqq T_{s_1}T_{s_2}\cdots T_{s_m}$.

We write $\tau : \mathcal{H}_W \to A$ for the *trace* defined linearly by:

$$\tau(T_w^{-1}) \coloneqq \begin{cases} 1 & w = e, \\ 0 & w \neq e \end{cases} \qquad \text{for } w \in W.$$

THEOREM 3.3 ([20, Corollary 5.3]). *For any word* w *and* $u, v \in W$*, we have*

$$\left|R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q)\right| = q^{\ell(v)}\tau(T^{-1}_{v^{-1}}T_{\mathsf{w}}T^{-1}_{vu}).$$

4. Techniques, Examples, and Problems in Finite Coxeter Groups

In this section we describe some techniques and problems in the setting of finite reflection groups.

4.1. Technique: traces and Schur elements. Let W be a finite Coxeter group and $\mathrm{Irr}(W)$ be the set of its irreducible characters. The Hecke algebra $\mathcal{H}_W$ decomposes similarly to the group algebra of W, with certain weights called *Schur elements* $\mathbf{s}_\tau(\chi_q)$:

$$\tau = \sum_{\chi\in\mathrm{Irr}(W)} \frac{1}{\mathbf{s}_\tau(\chi_q)}\chi_q. \tag{4.1}$$

This decomposition of $\mathcal{H}_W$, along with explicit identifications of the Schur elements gives a general technique for computing $\left|R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q)\right|$ in a case-by-case manner for any finite Coxeter group by taking a sum over irreducible characters. Furthermore, when taking a union of braid varieties $R^{(v)}_{e,\mathsf{w}}$ over all elements $v \in W$, we have the following dramatic simplification of Theorem 3.3.

THEOREM 4.1 ([20, Section 6.6]). *For* W *finite,*

$$\sum_{v\in W}\left|R^{(v)}_{e,\mathsf{w}}(\mathbb{F}_q)\right| = \sum_{\chi\in\mathrm{Irr}(W)} \dim(\chi)\cdot\chi_q(T_{\mathsf{w}}).$$

There are many interesting examples using these two techniques, as we now discuss.

4.2. Example: Galashin and Lam Positroid Catalan Combinatorics. In [18, 19], Galashin and Lam apply this framework in the context of type A rational Catalan combinatorics. We refer the reader to their excellent papers for a full description of their results and open problems.

4.3. Example: Rational Noncrossing Coxeter–Catalan Combinatorics. In [20], we solved two long-standing open problems in Coxeter–Catalan combinatorics using this framework. (There are still many open combinatorial problems stemming from this work.) For any positive integer p coprime to h, we set

$$\mathsf{Cat}_p(W;q) \coloneqq \prod_{i=1}^{r}\frac{[p+(pe_i \bmod h)]}{[d_i]}, \tag{4.2}$$

where $d_1 \leq d_2 \leq \cdots \leq d_r$ are the *degrees* of W, $e_i = d_i - 1$ are the exponents, h is the Coxeter number, and where $0 \leq (pe_i \bmod h) < h$ is the integer in that range congruent to pe_i modulo h. (For well-generated finite complex reflection groups, $\mathsf{Cat}_p(W;q)$ is the graded character of the finite-dimensional irreducible representation $eL_{p/h}(\mathsf{triv})$ of the rational Cherednik algebra at the parameter p/h). We write $\mathrm{Cat}(W) \coloneqq \mathrm{Cat}_{h+1}(W;1)$, which was previously known case-by-case to enumerate the noncrossing partitions. Two long-standing open problems in Coxeter–Catalan combinatorics were:

- uniformly prove that noncrossing objects are enumerated by $\mathrm{Cat}(W)$.
- uniformly construct rational noncrossing objects enumerated by $\mathrm{Cat}_p(W;1)$.

Our framework recently led to a solution to both of these problems in [20], along with their parking analogues.

THEOREM 4.2 ([20]). *For any (finite) Weyl group W of rank r and Coxeter number h, Coxeter word c, and integer p coprime to h, we have*

$$|R_{e,\mathsf{c}^p}(\mathbb{F}_q)| = (q-1)^r \mathsf{Cat}_p(W;q) \text{ and } \sum_{v\in W} \left|R^{(v)}_{e,\mathsf{c}^p}(\mathbb{F}_q)\right| = (q-1)^r [p]^r.$$

Sending $q \to 1$, we conclude that for any (irreducible, finite) Coxeter group W of rank r and Coxeter number h, Coxeter word c, and (positive) integer p coprime to h, we have $|\mathcal{M}_{e,\mathsf{c}^p}(W)| = \mathsf{Cat}_p(W)$ and $\sum_{v\in W} \left|\mathcal{M}^{(v)}_{e,\mathbf{c}^p}\right| = p^r$. We showed that our maximal distinguished subwords are truly *noncrossing* by giving a natural uniform bijection between $\mathcal{M}_{e,\mathsf{c}^p}(W)$ and noncrossing partitions for $p = mh+1$ [1, 39], and between $\bigcup_{v\in W} \mathcal{M}^{(v)}_{e,\mathbf{c}^p}$ and Armstrong, Reiner, and Rhoades's noncrossing parking functions for $p = h+1$ [5]. In particular, our results give the first uniform proof that the number of clusters in a finite-type cluster algebra is counted by $\mathsf{Cat}(W)$ [17, Theorem 1.9].

PROBLEM 4.3. *Develop graphical models for our rational noncrossing Catalan and parking objects (generalizing the usual depictions of noncrossing partitions in classical types).*

One possible approach in the symmetric group would be to attempt to match $\mathcal{M}_{e,\mathsf{c}^p}(W)$ up with existing models for rational noncrossing partitions [3, 10].

Beyond the maximal distinguished subwords, enumerative results are lacking (and the non-maximal distinguished subwords do not appear to be closely linked to the h- or f-vector of the associahedron). Preliminary calculations in type A suggest that these lower order terms have a deeper, perhaps not unexpected, connection to maps—just as there are the same number of maximal distinguished subwords and rooted bicolored unicellular maps of genus 0 on n edges, there appear to be the same number of distinguished subwords with $r+2$ skips as rooted bicolored unicellular maps of genus 1 on $n+2$ edges (given by $\frac{(2n+3)!}{6n!(n+1)!}$).

4.4. Problem: Rational Nonnesting Catalan Combinatorics. W-Catalan numbers naturally appear in a markedly different context—in the study of affine Weyl groups and affine Springer fibers. Specializing to crystallographic Coxeter groups, for p coprime to h, $\mathsf{Cat}_p(W)$ (uniformly) counts the number of coroot points inside a p-fold dilation of the fundamental alcove in the corresponding affine Weyl group [26, 41]. For $p = h+1$, these coroot points are called *nonnesting partitions*, and are in bijection with order ideals in the root poset (or, equivalently, ad-nilpotent ideals in a Borel subalgebra of the corresponding complex simple Lie algebra). *Rational (nonnesting) parking functions* are the p^n alcoves inside this same dilation of the fundamental alcove.

Although nonnesting and noncrossing partitions have many similarities, finding a uniform bijection between the two sets has been an active and motivating area of research since the late 1990s [37, 7]. The state of the art has now changed with our recent new definition of rational noncrossing objects—both noncrossing and nonnesting objects are finally defined at almost the same level of generality: both are defined for Weyl groups and for any p coprime to h.

PROBLEM 4.4. *Let p be coprime to the Coxeter number h.*

- *Find a bijection between $\mathcal{M}_{e,\mathsf{c}^p}$ and rational nonnesting partitions.*
- *Find a bijection between $\bigcup_{v\in W} \mathcal{M}^{(v)}_{e,\mathbf{c}^p}$ and rational nonnesting parking functions.*

In [44] and for $p = h+1$, we conjectured exactly such a bijection between nonnesting and noncrossing objects for any Coxeter element and any finite Weyl group, suggesting that the root poset encodes a remarkable amount of information related to the corresponding Weyl group (compare with the duality between the heights of roots and the degrees). Our conjectural bijection between noncrossing and nonnesting objects comes from mimicking walks on the W-associahedron—drawing inspiration from [36, 9, 6], our methods produce remarkable conjectural (compatible) bijections from nonnesting partitions to clusters and noncrossing partitions which have been exhaustively checked up to rank eight [44, 45, 40].

PROBLEM 4.5. *Show the conjectured maps in [44, 45] are bijections between nonnesting and noncrossing partitions. Use the new definition of rational noncrossing objects from [20] to extend them to the Fuss and rational levels of generality.*

With a group at LaCIM consisting of Dequêne, Frieden, Iraci, Schreier-Aigner, and Thomas, we have recently made substantial progress on this problem, proving part of the conjectures by purely combinatorial means in type A, for all Coxeter elements [15]. The innovation is that we are able to find an element that realizes the Cambrian recurrence on nonnesting partitions—a similar approach might work for other types.

4.5. Problem: Periodic elements. While Equation (4.1) enables brute-force computations of any traces for finite Coxeter groups, there are also more powerful specific tools available for sufficiently nice words w (a special case of which was used in Section 4.3.

We say that a braid $\mathsf{w} = \mathbf{s}_1 \cdots \mathbf{s}_{\ell(\mathsf{w})} \in B^+(W)$ is *periodic* if $\mathsf{w}^m = \mathsf{w}_\circ^{2p}$ for some p, m with $m \neq 0$. There a classification of periodic braids up to conjugacy using Springer theory as the dth roots of the full twist, where d is a regular number of W [8, 31, 21]: in type A we have only (conjugates of) $\mathbf{c} = \mathbf{s}_1 \cdots \mathbf{s}_n$ and $(\mathbf{s}_1 \cdots \mathbf{s}_n)\mathbf{s}_1$; in type D we have $\mathbf{c} = \mathbf{s}_1 \cdots \mathbf{s}_n$ and $(\mathbf{s}_1 \cdots \mathbf{s}_{n-2})\mathbf{s}_{n-1}(\mathbf{s}_1 \cdots \mathbf{s}_{n-2})\mathbf{s}_n$.

PROBLEM 4.6. *Find formulas for $|R_{e,\mathsf{w}^p}(\mathbb{F}_q)|$ for powers of periodic elements w^p. Extend to formulas for $\sum_{v\in W} \left|R^{(v)}_{e,\mathsf{w}^p}(\mathbb{F}_q)\right|$.*

Small computations suggest that there should be a uniform formula, generalizing the usual Coxeter–Catalan numbers, using the regular number d, the eigenvalues of the periodic element, and a subset of the degrees—such a formula should immediately follow after tracing through the method outlined below, taking advantage of formulas for character values of periodic elements. The extension to the sum should be straightforward using Theorem 4.1.

EXAMPLE 4.7. For type D_4 with $d = 4$ and $\mathsf{w} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_4$, we have that

$$\left|R_{e,\mathsf{w}^3}(\mathbb{F}_q)\right| = q^{-18}(q-1)^4(1+q^2+3q^4+4q^6+4q^8+3q^{10}+q^{12}+q^{14}), \text{ and}$$

$$\sum_{v\in W} \left|R^{(v)}_{e,\mathsf{w}^3}(\mathbb{F}_q)\right| = q^{-18}(q-1)^4(1+q+q^2)^4(1+4q^3+q^6).$$

At $q=1$ and for p odd, we appear to have $\lim_{q\to 1}(q-1)^{-4}\left|R_{e,\mathsf{w}^p}(\mathbb{F}_q)\right| = \frac{((p+1)(p+3))^2}{32}$; note that the order d of w is 4, and that the eigenvalues of $w = s_1s_2s_3s_1s_2s_4$ in the reflection representation are i^1 and i^3 (each with multiplicity 2).

Problem 4.8. *Develop graphical models for the maximal distinguished subwords $\mathcal{M}_{e,\mathsf{w}^p}$ (generalizing the usual depictions of noncrossing partitions in classical types).*

4.6. Technique: Periodic elements. There is a concrete, systematic, and uniform approach to Problem 4.6, building on [**20**]. For all $\chi \in \mathrm{Irr}(W)$, the *fake* and *generic degrees* of χ are

$$\mathrm{Feg}_\chi(q) := \frac{(\chi, [\mathfrak{S}]_q)_W}{(1, [\mathfrak{S}]_q)_W} \text{ and } \mathrm{Deg}_\chi(q) := \frac{\mathbf{s}^+(1_q)}{\mathbf{s}^+(\chi_q)}.$$

It turns out that $\mathrm{Feg}_\chi(q) \in \mathbb{Z}[q]$ and $\mathrm{Deg}_\chi \in \mathbb{Q}_W[q]$; at $q=1$, both polynomials specialize to the *degree* of χ. For W a finite Coxeter group and $\chi \in \mathrm{Irr}(W)$ an irreducible character, write $c(\chi) = \frac{1}{\dim(\chi)}\sum_{t\in T}\chi(t)$ for the *content* of χ [**42**]. (When $W = \mathfrak{S}_{n+1}$ is the symmetric group and the irreducible characters are indexed by integer partitions, this agrees with the usual definition of content (as the sum of the contents of all boxes in the partition.) Then we have the following result on traces of periodic elements [**42**]: if $\chi \in \mathrm{Irr}(W)$ and w is a periodic braid of slope $\nu \in \mathbb{Q}$, write $\sigma_\mathsf{w} = q^{-\ell(\mathsf{w})/2}T_\mathsf{w}$. Then $\chi_q(\sigma_\mathsf{w}) = q^{\nu\, c(\chi)}\mathrm{Feg}_\chi(e^{2\pi i\nu})$ so that

$$\begin{aligned}\tau(\sigma_\mathsf{w}) &= \frac{\varepsilon(w)}{\mathbf{s}^+(1_q)}\sum_{\chi\in\mathrm{Irr}(W)} q^{-\nu\, c(\chi)}\mathrm{Feg}_\chi(e^{2\pi i\nu})\mathrm{Deg}_\chi(q)\\ &= \frac{\varepsilon(w)}{\mathbf{s}^+(1_q)}\sum_{\chi\in\mathrm{Irr}(W)} q^{-\nu\, c(\chi)}\mathrm{Feg}_\chi(q)\mathrm{Deg}_\chi(e^{2\pi i\nu}),\end{aligned}$$

where the second equality follows from Lusztig's exotic Fourier transform. This last formula gives a dramatic simplification of the trace—even though periodic elements often have many vanishing characters, transferring the root of unity from the generic degree to the fake degree typically gives a dramatic reduction in the number of terms vanishing in the sum.

It follows from the above (see also [**42**, Theorem 9.2.1] and Theorem 4.1) that

$$\sum_{v\in W}\left|R^{(v)}_{e,\mathsf{w}_\circ^2}(\mathbb{F})\right| = \sum_{\chi\in\mathrm{Irr}(W)}\dim(\chi)^2 q^{\mathrm{cont}(\chi)}.$$

More generally, writing $\mathrm{ord}(w)$ for the order of an element $w \in W$, the expression $\sum\limits_{v\in W}\left|R^{(v)}_{e,\mathsf{w}^{\mathrm{ord}(w)}}(\mathbb{F})\right|$ appears to have positive coefficients.

4.7. Problem: Complex reflection groups. In this section we propose an interesting extension of the results in [**20**] to *spetsial* complex reflection groups (that is, $G(d,1,r)$, $G(d,d,r)$, G_4, G_6, G_8, G_{14}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, G_{28}, G_{29}, G_{30}, G_{32}, G_{33}, G_{34}, G_{35}, G_{36}, or G_{37}). Spetsial complex reflection groups still have a preferred set of "simple reflections", Coxeter elements, and a well-defined rational Catalan number [**22**]. Moreover, the notion of periodic elements naturally generalizes to well-generated complex reflection groups, and such elements have been classified. We may therefore consider Problem 4.6 in this context.

The first step in this direction is to (subject to certain assumptions common in the field) compute the trace $\tau(T_{\mathbf{c}}^p)$ in the Hecke algebra of the well-generated complex reflection group [**11, 12**] (whose braid groups have Artin-like presentations, by Bessis); many of the favorable representation-theoretic properties of Coxeter elements that we used in Section 4.3 still carry over to the complex setting [**11**]. While we no longer have Lusztig's exotic Fourier transformation uniformly or a uniform definition of the Hecke algebra, Trinh has pointed out that there are partial results due to Lasy and Lacabanne in the infinite family [**30, 28**], and a reciprocity result due to Malle that has been used by Douvropoulos in a related setting [**16**]. In any event, the computation can be carried out case-by-case using Schur elements and the corresponding decomposition of the Hecke algebra.

Conjecture 4.9. *Let W be a spetsial complex reflection group. Then (up to a power of q) we have*

$$\tau(T_{\mathbf{c}}^p) = (q-1)^r \prod_{i=1}^{r} \frac{[p+e_i(V^p)]}{[d_i]},$$

where the $e_i(V^p)$ are the fake degrees of the p-th Galois twist of the reflection representation and the trace is taken in the Hecke algebra $\mathcal{H}_W$.

As part of his undergraduate honors thesis, Weston Miller has confirmed Conjecture 4.9 on all exceptional spetsial complex reflection groups; moreover, Conjecture 4.9 is false for non-spetsial well-generated groups.

Example 4.10. The complex reflection group G_4 has rank $r = 2$, Coxeter number $h = 6$. Its reflection representation has fake degrees 3 and 5. We computed using GAP3 that

$$\tau(T_{\mathbf{c}}^7) = (q-1)^2(q^{12}+q^8+q^6+q^4+1) = (q-1)^2\frac{[7+3][7+5]}{[4][6]}.$$

The Deodhar decomposition allows us to build combinatorial models of braid varieties for general Coxeter groups. Besides the representation-theoretic computation above, there is the issue of finding the correct combinatorial definition of distinguished subwords for complex reflection groups.

Problem 4.11. *Find a combinatorial description of the Deodhar decomposition for well-generated complex reflection groups.*

5. Techniques, Examples, and Problems in Affine Weyl Groups

In this section we work with affine Weyl groups $\widetilde{W}$.

5.1. Technique: Opdam's trace formula. Write Φ^+ for the positive roots of a simple Lie group, $Q = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ for the *root lattice*, $Q^+ \subset Q$ for the positive span of the simple roots, and Λ for the weight lattice. Given $\lambda \in Q^+$, we express λ in the basis of fundamental weights as $\lambda = \sum_{i=1}^{n-1} a_i\lambda_i$ and define $\lambda_+ = \sum_{i:a_i>0} a_i\lambda_i$ and $\lambda_- = -\sum_{i:a_i<0} a_i\lambda_i$. For $x \in \Lambda$, we write t_x for the translation in the extended affine Weyl group $\widehat{W}$.

Definition 5.1. A *Kostant partition* $(a_\alpha)_{\alpha\in\Phi^+}$ for $\lambda \in Q^+$ is a sequence of nonnegative integers indexed by positive roots such that $\lambda = \sum_{\alpha\in\Phi^+} a_\alpha\alpha$. We denote the set of all Kostant partitions for λ by $K(\lambda)$.

Opdam proved the following trace formula, which—when combined with Theorem 3.3—is our main technique in this setting.

THEOREM 5.2 ([**35**, Cor. 1.18]). *Let* $[k]_q = \frac{(q-1)^2}{q}\frac{q^k-q^{-k}}{q-q^{-1}}$. *For* $\lambda \in Q^+$,

$$\tau(T_{t_{\lambda_-}}T^{-1}_{t_{\lambda_+}}) = q^{(\ell(t_{\lambda_-})-\ell(t_{\lambda_+}))/2}\sum_{(a_\alpha)\in K(\lambda)}\prod_{\substack{\alpha\in\Phi^+\\ a_\alpha>0}}[a_\alpha]_q.$$

5.2. Example: Haglund's formula and Tesler matrices. The results in this section were obtained in collaboration with Galashin and Lam. Let S_n act diagonally on the polynomial ring $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$, and write Q_n for its root lattice, Λ_n for its fundamental weights, and Φ_n^+ for its positive roots. The *quotient ring of diagonal coinvariants* DH_n is the quotient of this polynomial ring by the ideal generated by the invariants with no constant term; there is a more general S_n module DH_n^m depending on an integral parameter m. In [**24**], Haglund proved a remarkable formula for the bigraded (in x- and y-degree) Hilbert series of DH_n^m. Haglund stated the formula in terms of *Tesler matrices*, which are a simple combinatorial rephrasing of Kostant partitions. We choose to write the formula using Kostant partitions to mirror Theorem 5.2—note that since there are only $n-1$ simple roots in Φ_n^+, the formula for DH_{n-1}^m is written using Kostant partitions in Q_n^+.

THEOREM 5.3 ([**24**, Corollary 1 and Theorem 3]). *Write* $[k]_{q,t} = (q-1)(1-t)\frac{q^k-t^k}{q-t}$ *and let* $\lambda = (m(n-1)+1)\lambda_{n-1} - (m-1)\lambda_1 \in Q_n^+$. *Then*

$$\mathrm{Hilb}(\mathrm{DH}_{n-1}^m;q,t) = \left(\frac{1}{(q-1)(t-1)}\right)^{n-1}\sum_{(a_\alpha)\in K(\lambda)}\prod_{\substack{\alpha\in\Phi_n^+\\ a_\alpha>0}}[a_\alpha]_{q,t}.$$

Let S_n be the symmetric group of order $n!$. The Weyl group of $\mathrm{GL}_n(\mathbb{F})$ is the group of *extended affine permutations* $\widehat{S}_n = \Lambda_n \ltimes S_n \simeq N_{\mathrm{GL}_n(\mathbb{F})}(\widehat{T})/\widehat{T}$, whose elements can be thought of as bijections $\widehat{w}:\mathbb{Z}\to\mathbb{Z}$ such that $\widehat{w}(i+n) = \widehat{w}(i)+n$ and $\sum_{i=1}^n \widehat{w}(i) = \binom{n+1}{2} \bmod n$. Given $\lambda \in Q_n^+$, we express λ in the basis Λ_n as $\lambda = \sum_{i=1}^{n-1} a_i\lambda_i$ and define $\lambda_+ = \sum_{i:a_i>0} a_i\lambda_i$ and $\lambda_- = -\sum_{i:a_i<0} a_i\lambda_i$. For $x \in \Lambda_n$, we write t_x for the corresponding translation element of $\widehat{S}_n$.

EXAMPLE 5.4. The element $\lambda = \alpha_1 + 3\alpha_2 + 5\alpha_3 \in Q_4^+$ has ten possible Kostant partitions, partially ordered in Figure 1 according to the PBW basis terms appearing in the corresponding element of the canonical basis in the positive part of the quantum group $U_q^+(\mathfrak{sl}_4)$ (for the ordering $\alpha_1 < \alpha_{12} < \alpha_{123} < \alpha_2 < \alpha_{23} < \alpha_3$; it would be interesting to study this partial ordering in more detail; see also [**4, 34**]). Since $\lambda = -\lambda_1 + 7\lambda_3$, we have $\lambda_+ = 7\lambda_3$ and $\lambda_- = \lambda_1$, so that $t_{\lambda_1} = \sigma\cdot s_3s_2s_1$ and $t_{7\lambda_3} = (\sigma^3\cdot s_1s_2s_3)^7$.

EXAMPLE 5.5. As in Example 5.4 and Figure 1, $\lambda = -\lambda_1 + 7\lambda_3$ has ten Kostant partitions, while $\ell(t_{\lambda_1}) = 3$ and $\ell(t_{7\lambda_3}) = 21$. Taking the weighted sum over $K(\lambda)$, we have $\mathrm{tr}(T_{t_{\lambda_-}}T^{-1}_{t_{\lambda_+}}) =$

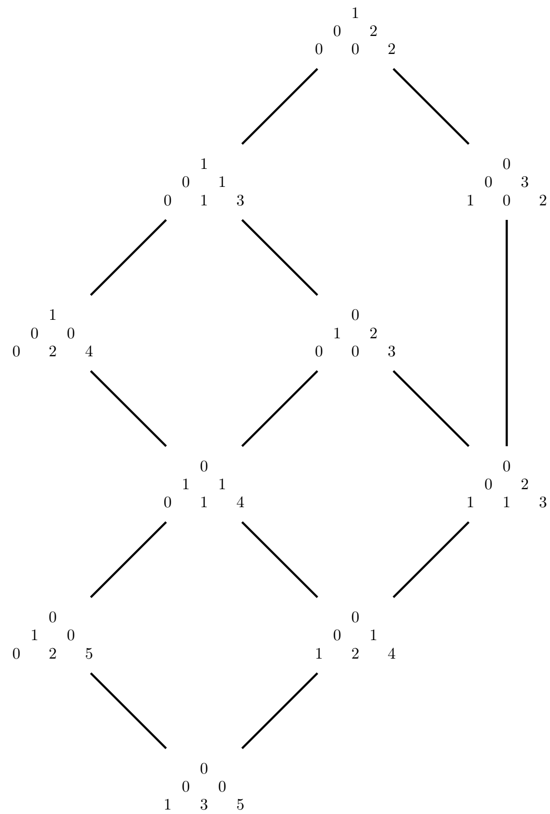

FIGURE 1. The set of Kostant partitions $K(\alpha_1 + 3\alpha_2 + 5\alpha_3) \in Q_4^+$, partially ordered using the canonical basis of the quantum group $U_q^+(\mathfrak{sl}_4)$. Coefficients are arranged by dominance ordering on positive roots.

$$
\begin{aligned}
&q^{(3-21)/2}\big([1]_q[3]_q[5]_q + [1]_q[2]_q[1]_q[4]_q + [1]_q[1]_q[2]_q[3]_q + [1]_q[3]_q[2]_q + [1]_q[2]_q[5]_q + \\
&\quad + [1]_q[1]_q[1]_q[4]_q + [1]_q[2]_q[3]_q + [2]_q[1]_q[4]_q + [1]_q[1]_q[1]_q[3]_q + [1]_q[2]_q[2]_q\big) \\
&= q^{-18}(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)^2(q-1)^6
\end{aligned}
$$

Since $[k]_q = [k]_{q,q^{-1}}$, we can specialize Theorem 5.3 using a result of Haiman to conclude the following.

THEOREM 5.6. *Fix the extended affine Weyl group $\widehat{S}_n$, and let $v = t_{(m-1)\lambda_1}$ and $w = t_{(m(n-1)+1)\lambda_{n-1}}$. Then*

$$|R_{v,\mathsf{w}}(\mathbb{F})| = (q-1)^{2(n-1)}\mathrm{Hilb}(\mathrm{DH}^m_{n-1}; q, q^{-1}).$$

PROOF. By [**27**, Lemmas A3 and A4] and [**20**, Corollary 5.3], the number of $\mathbb{F}_q$-points in the braid variety $R_{\widetilde{v},\widetilde{w}}(\mathbb{F})$ is given by the trace formula $q^{\ell(\widetilde{w})-\ell(\widetilde{v})}\mathrm{tr}(T_{\widetilde{v}}\, T_{\widetilde{w}}^{-1})$.

Since $\widehat{v} = t_{(m-1)\lambda_1}$ and $\widehat{w} = t_{(m(n-1)+1)\lambda_n}$, we see that

$$\widehat{v} = (\sigma s_{n-1} \cdots s_2 s_1)^{m-1} = \widetilde{v}\sigma^{m-1} \text{ and}$$
$$\widehat{w} = (\sigma^{-1} s_1 \cdots s_{n-2} s_{n-1})^{m(n-1)+1} = \widetilde{w}\sigma^{-m(n-1)-1} = \widetilde{w}\sigma^{-mn+(m-1)} = \widetilde{w}\sigma^{m-1},$$

so that $T_{\widehat{v}} = T_{\widetilde{v}} T_{\sigma^{m-1}}$ and $T_{\widetilde{w}}^{-1} = T_{\sigma^{m-1}}^{-1} T_{\widetilde{w}}^{-1}$. We conclude that $\mathrm{tr}(T_{\widehat{v}}\, T_{\widehat{w}^{-1}}) = \mathrm{tr}(T_{\widetilde{v}}\, T_{\widetilde{w}^{-1}})$, so that $\mathrm{tr}(T_{\widetilde{v}}\, T_{\widetilde{w}^{-1}})$ is given by Opdam's Theorem 5.2. We conclude the result by Haglund's Theorem 5.3. □

Example 5.7. The 16 maximal Deodhar words for $\widetilde{S}_4$ and $\mathcal{M}_{e,(s_0s_1s_2s_3)^3}$ are given in Figure 2; the word $(s_0s_1s_2s_3)^3$ is drawn on three lines, and the letters appearing in a subword are highlighted in gray.

Problem 5.8.

- *Find a bijection between $\mathcal{M}_{e,t_{n\lambda_{n-1}}}$ in $\widetilde{S}_n$ and (noncrossing or nonnesting) parking functions for S_{n-1}.*
- *Extend to $\mathcal{M}_{t_{(m-1)\lambda_1}, t_{(m(n-1)+1)\lambda_{n-1}}}$ and Fuss parking functions.*

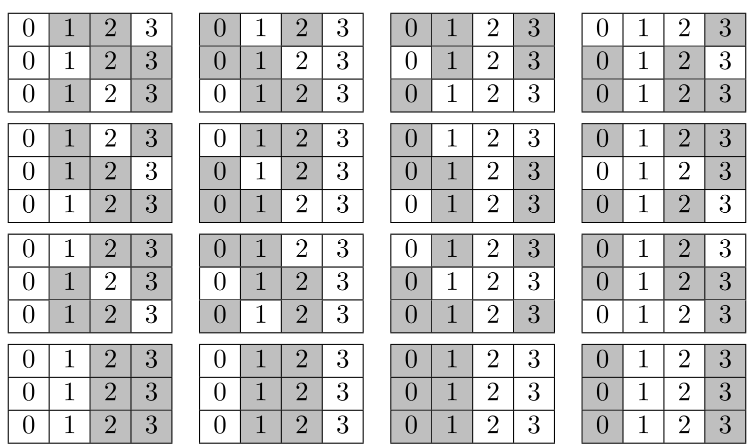

Figure 2. The 16 maximal distinguished subwords for $\widetilde{S}_4$ and $\mathcal{M}_{e,(s_0s_1s_2s_3)^3}$; the letters appearing in a subword are highlighted in gray.

It appears that nice enumerations extend to other fundamental weights beyond λ_{n-1}. It should be possible evaluate the sum over the Kostant partitions using inductive methods similar to those Haglund used to establish Theorem 5.3.

5.3. Problems: affine Weyl groups. Galashin has pointed out that there is a similar formula due to Gorsky and Negut [**23**], which appears as [**25**, Theorem 32]—this formula sums over the same set of Kostant partitions and outputs the same q,t-polynomial, but extends to the rational case. We propose an extension of Theorem 4.1 to rational parking functions.

Conjecture 5.9. *Let n and m be relatively prime, and define the element*

$$c_{nm} = (s_{m-n+1}s_{m-n+2}\cdots s_{m-1}s_0s_{m-n}s_{m-n-1}\cdots s_1)^n \in \widetilde{S}_m.$$

Then
$$|R_{e,c_{nm}}(\mathbb{F})| = (q-1)^{n+m-1}[m]^{n-1}.$$

This agrees with the computatations in Section 5.2 for $(n,m) = (n,n+1)$, since it is easily seen that $w_{n,n+1}$ produces a braid variety isomorphic to the one given by $t_{n\lambda_{n-1}}$. We have confirmed Conjecture 5.9 by computer for $(n,m) \in \{(3,5),(3,8),(4,7),(5,7)\}$; note that Opdam's formula does not apply because we don't get cancellation of the non-translation elements in this more general case. One can again ask for bijections between $\mathcal{M}_{e,c_{nm}}$ and rational parking functions.

The following conjecture is a restatement of a conjecture of Armstrong, Garsia, Haglund, Rhoades, and Sagan in the language of Kostant partitions and our framework, and would generalize Theorem 4.1.

Conjecture 5.10 ([**2**, Conjecture 7.1]). *Fix $\widetilde{W} = \widetilde{S}_n$. Let $\lambda \in Q^+$ satisfy $\lambda = \sum_{i=1}^{n-1} a_i\alpha_i = t_{\lambda_+} - t_{\lambda_-}$ with $a_1 > a_2 > \cdots > a_{n-1} \geq a_n = 0$. Then*
$$\left|R_{t_{\lambda_-},t_{\lambda_+}}(\mathbb{F})\right| = q^{(\ell(t_{\lambda_+})-\ell(t_{\lambda_-}))/2}(q-1)^{2n}\prod_{i=1}^{n-1}[(i+1)a_i - ia_{i+1}].$$

While we can express the left-hand side of the conjecture as a sum of Kostant partitions, the form of the right-hand side strongly suggests that there is another decomposition of the variety available. It would be interesting to try to extend the previous conjecture to other affine Weyl groups, and other affine Weyl group elements other than translations.

Write f_μ for the number of standard Young tableaux of shape μ.

Conjecture 5.11. *Fix $\widetilde{W} = \widetilde{S}_{2n}$ and take $2\lambda_n \in Q^+$. Then*
$$\left|R_{1,t_{2\lambda_n}}(\mathbb{F})\right| = \sum_{(a_\alpha)\in K(2\lambda_n)}\prod_{\substack{\alpha\in\Phi^+\\ a_\alpha>0}}[a_\alpha]_q = \sum_{\mu\vdash n} f_\mu^2 q^{2\left(c(\mu)+\binom{n}{2}\right)}.$$

We have confirmed Conjecture 5.11 up to $n = 5$. To use Opdam's formula, we need to compute the sum over $K(2\lambda_n)$—the size of $|K(2\lambda_n)|$ starts $1, 5, 86, 4274, 550919, \ldots$, and does not appear in the OEIS.

Example 5.12. The six maximal Deodhar words for $\widetilde{S}_5$ and $\mathcal{M}_{e,2t_{\lambda_3}}$ are illustrated in Figure 3. A reduced word for $2t_{\lambda_3}$ is $[s_0,s_1,s_2,s_5,s_0,s_1,s_4,s_5,s_0,s_3,s_4,s_5,s_2,s_3,s_4,s_1,s_2,s_3]$, which we have split into two 3×3 squares—the left square containing the first nine letters and the right square containing the remaining nine letters. If one interprets the shaded letters as '0' and the unshaded letters as '1', then each pair of 3×3 squares appears to be the permutation matrix of a permutation, along with the permutation matrix of its inverse.

More generally, one could consider the same problem for $ma\lambda_b \in Q^+$ in type $\widetilde{S}_{ab}$.

Conjecture 5.13. *Fix $\widetilde{W}$ to be the affine Weyl group of type C_n and $w = (s_0s_1\cdots s_n)^{2k+1}$. Then there exists a statistic* stat *on the set of lattice points $L_{k,n} = \{(x_1,\ldots,x_n)\in\mathbb{Z}^n : |x_1|+\cdots+|x_n| \leq k\}$ such that*
$$\pm\left|R_{e,w}(\mathbb{F}_q)\right|_{q\mapsto -q} = (q+1)^{n+1}\sum_{(x_1,\ldots,x_n)\in L_{k,n}} q^{\mathrm{stat}(x_1,\ldots,x_n)}$$

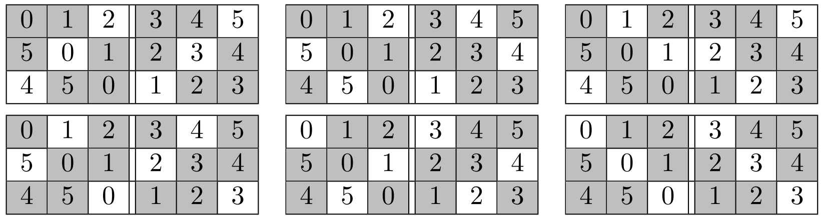

FIGURE 3. The six maximal Deodhar words for $\widetilde{S}_5$ and $\mathcal{M}_{e,2t_{\lambda_3}}$.

This has been confirmed computationally in some small cases; one can apply Opdam's formula to the cases when $n|(2k+1)$. By [**13**], the number of such lattice points is $\sum_{i=0}^{n} 2^i \binom{n}{n-i}\binom{k}{i}$. For fixed n, the generating function for k is given by $\frac{(1+q)^n}{(1-q)^{n+1}}$.

6. Techniques and Problems in Kac-Moody groups

In this section, we fix a general Kac–Moody Lie group G and describe general techniques that apply.

6.1. R-polynomials. Let W be the Weyl group of a Kac-Moody Lie group. There are two general technique that persist at this level of generality. The first exploits a recursive structure on the set of distinguished subwords to construct polynomials that count $\left|R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q)\right|$: setting $R^{(v)}_{u,\varnothing}(q) := R_{u,\varnothing}(q)$, for any word w and $s \in S$ we have

$$R^{(v)}_{u,\mathsf{ws}}(q) = \begin{cases} R^{(v)}_{us,\mathsf{w}}(q) & \text{if } vus < vu \\ qR^{(v)}_{us,\mathsf{w}}(q) + (q-1)R^{(v)}_{u,\mathsf{w}}(q) & \text{if } vus > vu. \end{cases} \tag{6.1}$$

THEOREM 6.1. *For arbitrary Weyl groups W, we have* $R^{(v)}_{u,\mathsf{w}}(q) = \left|R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q)\right|$.

Braid varieties arising from powers of Coxeter elements are likely varieties of interest for general W, and serve as analogues of rational noncrossing Catalan objects for general Coxeter groups.

6.2. Cluster Varieties. Having identified interesting varieties, the second technique is a cluster-theoretic approach for computing the mixed Hodge decomposition. For particular choices of u and w, it is possible to put a (locally acyclic) cluster structure on $R^{(v)}_{u,\mathsf{w}}(\mathbb{C})$. With this cluster structure, we may invoke technology of Lam and Speyer [**29, 33**] to recursively compute $|R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q)|$. This requires finding an artful sequence of mutations to isolate a separating edge, thereby enabling recursive arguments. This technique succeeds in small examples, but we require more systematic approaches using the specific properties of the quivers corresponding to the braid varieties of interest to handle large cases.

7. Cohomology and Mixed Hodge Structures

This section is speculative. Working over $\mathbb{C}$ now (rather than $\mathbb{F}$), we have the *Deligne splitting* of cohomology

$$H^k(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = \bigoplus_{p=0}^{k}\bigoplus_{q=0}^{k} H^{k,(p,q)}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})).$$

Other than the long exact sequence for relative cohomology, we have only one tool. For particular choices of u and w—even, for example, in affine type—it is possible to put a (locally acyclic) cluster structure on $R_{u,\mathsf{w}}^{(v)}(\mathbb{C})$, from which we may conclude that $H^{k,(p,q)}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = 0$ for $p \neq q$. With this cluster structure, we may invoke technology of Lam and Speyer [**38**], building on work of Muller [**33**]. As in the previous section, the approach is to find an artful sequence of mutations to isolate a separating edge. Following [**18**], when $R_{u,\mathsf{w}}^{(v)}(\mathbb{C})$ has dimension d we define the *mixed Hodge polynomial*

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}); q, t) \coloneqq \sum_{k,p\in\mathbb{Z}} q^{p-k/2} t^{(d-k)/2} \dim(H^{k,(p,p)}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))).$$

Problem 7.1. *Compute the mixed Hodge polynomial $\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))$ in all cases where the point count $|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)|$ has been established.*

This is already fascinating in the case of the Coxeter–Catalan varieties $R_{e,\mathbf{c}^p}$ and the parking varieties $\bigcup_{v\in W} R_{e,\mathbf{c}^p}^{(v)}$—here, the mixed Hodge polynomials should compute the rational (W, q, t)-analogues of Catalan numbers and parking functions [**22**].

Example 7.2. For $W = \mathfrak{S}_3$ and $\mathsf{w} = (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2)$, we have six varieties in $\bigsqcup_{v\in W} R_{e,\mathbf{c}^p}^{(v)}$ (one for each element v of W). Using the tables from [**29**], Lam has confirmed that as cluster varieties, they are of type E_6, A_4 (twice), A_2 (twice), and A_0, giving the sum

$$(q^3 + q^2t + qt + tq^2 + t^3) + 2(q^2 + qt + t^2) + 2(q + t) + 1,$$

which is the usual parking q, t-analogue of 4^2.

Problem 7.3. *Find combinatorial statistics* $\mathrm{stat}_q, \mathrm{stat}_t$ *on all distinguished subwords $\mathcal{D}_{u,\mathsf{w}}^{(v)}$ so that*

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = \sum_{\mathbf{u}\in\mathcal{D}_{u,\mathsf{w}}^{(v)}} [(q-1)(t-1)]^{\mathrm{e}_{\mathsf{u}}/2} q^{\mathrm{stat}_q(\mathbf{u})} t^{\mathrm{stat}_t(\mathbf{u})}.$$

When $\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))$ is a positive q, t-polynomial, find combinatorial statistics $\mathrm{stat}'_q, \mathrm{stat}'_t$ *on all maximal distinguished subwords $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ so that*

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = \sum_{\mathbf{u}\in\mathcal{M}_{u,\mathsf{w}}^{(v)}} q^{\mathrm{stat}'_q(\mathbf{u})} t^{\mathrm{stat}'_t(\mathbf{u})}.$$

Acknowledgements

It is an honor to be asked to contribute to this conference proceedings—especially as I am neither the most distinguished *Nathan*, nor the most distinguished *Williams* in algebraic combinatorics. Those in attendance for the final talk at Open Problems in Algebraic Combinatorics 2022 will observe that the content of this paper does not exactly match the content of that talk (slides are available at [46], a recording is available for streaming [47]).

I thank my coauthors Thomas Lam, Pavel Galashin, and Minh-Tâm Trinh for their very patient explanations and generous insights, as well as the three Dennises at the University of Minnesota (that is, Reiner, Stanton, and White) for their mentorship. I thank Carlos Arreche, Colin Defant, and Weston Miller for their helpful comments on a draft of this paper. I thank Ivan Marin for informative conversations on Hecke algebras of complex reflection groups.

References

1. D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. **202** (2009), no. 949, 159 pp.
2. D. Armstrong, A. Garsia, J. Haglund, B. Rhoades, and B. Sagan, *Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics*, 2011, pp. 451–494.
3. D. Armstrong, B. Rhoades, and N. Williams, *Rational associahedra and noncrossing partitions*, Electronic Journal of Combinatorics **20** (2013), no. 3.
4. Drew Armstrong, *Tesler matrices*, Talk slides at Bruce Saganfest, 2014.
5. Drew Armstrong, Victor Reiner, and Brendon Rhoades, *Parking spaces*, Advances in Mathematics **269** (2015), 647–706.
6. Drew Armstrong, Christian Stump, and Hugh Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Transactions of the American Mathematical Society **365** (2013), no. 8, 4121–4151.
7. Christos Athanasiadis, *On noncrossing and nonnesting partitions for classical reflection groups*, Electron. J. Combin **5** (1998), no. 42, 16.
8. David Bessis, *Finite complex reflection arrangements are $K(\pi,1)$*, Annals of Mathematics **181** (2006), no. 3, 809–904.
9. David Bessis and Victor Reiner, *Cyclic sieving of noncrossing partitions for complex reflection groups*, Ann. Comb. **15** (2011), 197–222.
10. Michelle Bodnar, *Rational noncrossing partitions for all coprime pairs*, J. Comb. **10** (2019), no. 2, 365–407.
11. M Broue, G Malle, and R Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, Journal fur die reine und angewandte Mathemaik **500** (1998).
12. Michel Broué, *Introduction to complex reflection groups and their braid groups*, Springer, 2010.
13. Daniel Bump, Kwok-Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler, *A local Riemann hypothesis, I*, Mathematische Zeitschrift **233** (2000), no. 1, 1–18.
14. Vinay V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. math. **79** (1985), no. 3, 499–511.
15. Benjamin Dequêne, Gabriel Frieden, Alessandro Iraci, Florian Schreier-Aigner, Hugh Thomas, and Nathan Williams, *Charmed roots and the Kroweras complement*, arXiv preprint arXiv:2212.14831 (2022).
16. Theo Douvropoulos, *On enumerating factorizations in reflection groups*, arXiv preprint arXiv:1811.06566 (2018).
17. Sergey Fomin and Andei Zelevinsky, *Y-systems and generalized associahedra*, Canad. Math. Bull. **45** (2002), no. 3, 537–566.
18. Pavel Galashin and Thomas Lam, *Positroids, knots, and q,t-Catalan numbers*, arXiv preprint arXiv:2012.09745 (2020).
19. ———, *Positroid Catalan numbers*, arXiv preprint arXiv:2104.05701 (2021).

20. Pavel Galashin, Thomas Lam, Minh-Tâm Quang Trinh, and Nathan Williams, *Rational Noncrossing Coxeter-Catalan Combinatorics*, arXiv preprint arXiv:2208.00121 (2022).
21. Owen Garnier, *Regular theory in complex braid groups*, arXiv preprint arXiv:2205.06383 (2022).
22. Iain Gordon and Stephen Griffeth, *Catalan numbers for complex reflection groups*, American Journal of Mathematics **134** (2012), no. 6, 1491–1502.
23. Eugene Gorsky and Andrei Negut, *Refined knot invariants and Hilbert schemes*, Journal de Mathématiques Pures et Appliquées **104** (2015), no. 3, 403–435.
24. James Haglund, *A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants*, Advances in Mathematics **227** (2011), no. 5, 2092–2106.
25. ———, *Catalan paths and q,t-enumeration*, Handbook of Enumerative Combinatorics (2015), 679–751.
26. Mark Haiman, *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin. **3** (1994), no. 1, 17–76.
27. David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Inventiones mathematicae **53** (1979), no. 2, 165–184.
28. Abel Lacabanne, *Fourier matrices for $G(d,1,n)$ from quantum general linear groups*, Journal of Algebra **586** (2021), 433–466.
29. Thomas Lam and David E Speyer, *Cohomology of cluster varieties, I: Locally acyclic case*, Algebra & Number Theory **16** (2022), no. 1, 179–230.
30. T Lasy, *Special Markov traces and Gomi's formula for reflection groups*, Ph.D. thesis, PhD thesis, University Denis Diderot, 2012.
31. Eon-Kyung Lee and Sang-Jin Lee, *Periodic elements in Garside groups*, Journal of Pure and Applied Algebra **215** (2011), no. 10, 2295–2314.
32. Bethany Marsh and Konstanze Rietsch, *Parametrizations of flag varieties*, Representation Theory of the American Mathematical Society **8** (2004), no. 9, 212–242.
33. Greg Muller, *$A = U$ for locally acyclic cluster algebras*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications **10** (2014), 094.
34. Jason O'Neill, *On the poset and asymptotics of Tesler matrices*, The Electronic Journal of Combinatorics **25** (2018), no. 2, 2–4.
35. Eric M Opdam, *A generating function for the trace of the Iwahori-Hecke algebra*, Studies in memory of Issai Schur, Springer, 2003, pp. 301–323.
36. Dmitri Panyushev, *On orbits of antichains of positive roots*, European Journal of Combinatorics **30** (2009), no. 2, 586–594.
37. Victor Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Mathematics **177** (1997), no. 1, 195–222.
38. David Speyer and Hugh Thomas, *Acyclic cluster algebras revisited*, Algebras, quivers and representations, Springer, 2013, pp. 275–298.
39. Christian Stump, Hugh Thomas, and Nathan Williams, *Cataland: Why the Fuss?*, arXiv preprint arXiv:1503.00710, 2015.
40. Christian Stump, Hugh Thomas, and Nathan Williams, *Cataland: Bijactions*, (2017).
41. Ruedi Suter, *The number of lattice points in alcoves and the exponents of the finite Weyl groups*, Mathematics of computation (1998), 751–758.
42. Minh-Tâm Quang Trinh, *From the Hecke category to the unipotent locus*, arXiv preprint arXiv:2106.07444v2 (2021).
43. Ben Webster and Milen Yakimov, *A Deodhar-type stratification on the double flag variety*, Transformation Groups **12** (2007), no. 4, 769–785.
44. N. Williams, *Cataland*, Ph.D. thesis, University of Minnesota, 2013.
45. Nathan Williams, *Bijactions in Cataland*, DMTCS Proceedings (2014), no. 01, 597–608.
46. ———, *Rational Noncrossing Coxeter-Catalan Combinatorics*, http://www.samuelfhopkins.com/OPAC/files/slides/n_williams.pdf, 2022, Slides; accessed: November 29, 2022.
47. ———, *Rational Noncrossing Coxeter-Catalan Combinatorics*, https://youtu.be/l7BVk4WiX9U, 2022, Recorded talk; accessed: November 29, 2022.

(N. Williams) University of Texas at Dallas
E-mail address: `nathan.williams1@utdallas.edu`