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Realizing simplicial complexes as the boundary of the totally non-negative part of a variety

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Abstract. We raise the question of when a simplicial complex can be recovered as the boundary of the totally non-negative part of a certain variety defined based on either the simplicial complex itself or on a "compatibility degree" function on pairs of vertices. We present some non-trivial examples with origins in combinatorics and representation theory, and briefly discuss the motivation from the study of scattering amplitudes.

1. The simplest form of the combinatorial problem

Let Δ be a simplicial complex on a ground set V. That is to say, we suppose that Δ is a collection of subsets of V, such that for any $F \in \Delta$, and $G \subseteq F$, we have $G \in \Delta$. We do not assume that all the elements of V appear in Δ.

Suppose further that Δ is *flag*, that is to say, the minimal non-faces of Δ are of size at most two. In other words, $F \in \Delta$ if and only if all the subsets of F of size at most two are in Δ.

Introduce a variable u_x for each $x \in V$, and consider the following system of equations, one for each $x \in V$:

$$u_x + \prod_{\{x,y\} \notin \Delta} u_y = 1 \tag{1.1}$$

Define $X(\Delta)$ to be the solutions to this system of equations over $\mathbb{C}$. Define $X_{\geq 0}(\Delta)$ to be the solutions with all u_x real and non-negative. We call this the totally non-negative part of $X(\Delta)$.

$\Sigma(\Delta)$ is a collection of subsets of V, describing the combinatorics of the boundary $X_{\geq 0}(\Delta)$, defined as follows. We say that $F \in \Sigma(\Delta)$ if there is some point $p \in X_{\geq 0}(\Delta)$ such that $u_x(p) = 0$ if and only if $x \in F$.

The question is: when does the combinatorics of the boundary of $X_{\geq 0}(\Delta)$ agree with Δ?

Problem 1.1. *For what V and Δ is it the case that $\Sigma(\Delta) = \Delta$?*

Let us look at some examples where this holds.

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Example 1.2. Let $V = \{1, 2\}$, and let $\Delta = \{\emptyset, \{1\}, \{2\}\}$. The two equations from (1.1) are both $x_1 + x_2 = 1$. The totally positive part of the solutions is a line segment, and the boundary recovers Δ.

Example 1.3. Let $V = \{1, 2, 3, 4, 5\}$. Let Δ be a five-cycle. It is less straightforward to show that $\Sigma(\Delta) = \Delta$.

Example 1.4. Fix a positive integer $N \geq 4$. Let V be the set of diagonals of an N-gon, and let Δ be the simplicial complex consisting of non-crossing collections of diagonals. The maximal faces of Δ then correspond to triangulations of the N-gon, and Δ is the (simplicial) associahedron. Then it is shown in [**3**] that $\Sigma(\Delta) = \Delta$. When $N = 4, 5$, this recovers the previous two examples.

We should remark that in general, we do not know that $X(\Delta)$ is reduced or irreducible, so the term "variety" in the title of this paper is intended in a loose sense.

2. A more complicated version of the combinatorial problem

As before, let V be a finite vertex set. Define a *compatibility degree* $c : V \times V \to \mathbb{Z}_{\geq 0}$ such that $c(x, y) = 0$ if and only if $c(y, x) = 0$. If $c(x, y) = 0 = c(y, x)$, we say that x and y are compatible.

Define $\Delta(c)$ to be the simplicial complex consisting of subsets of V which are pairwise compatible, and such that every element is compatible with itself.

Now consider the following system of equations, one for each $x \in V$:

$$u_x + \prod_{y \in V} u_y^{c(y,x)} = 1. \tag{2.1}$$

Write $X(c)$ for the solutions to (2.1), and similarly $X_{\geq 0}(c)$ and $\Sigma(c)$.

We recover the system (1.1) if we define the compatibility degree of x and y to be zero if $\{x, y\}$ is a face of Δ and 1 otherwise. The following is an example with a different choice of c.

Example 2.1. Let N be even, and consider the N-gon with the action of 180-degree rotation. Let $\widetilde{V}$ be the set of equivalence classes of diagonals under rotation. For $\widetilde{x}, \widetilde{y} \in \widetilde{V}$, define $\widetilde{c}(\widetilde{y}, \widetilde{x})$ to be the total number of intersections of all diagonals from $\widetilde{y}$ with one diagonal from $\widetilde{x}$. The faces of $\Delta(\widetilde{c})$ are in natural bijection with the partial triangulations of the N-gon which are symmetric under 180-degree rotation. (Note that this example of $\widetilde{c}$ does not satisfy $\widetilde{c}(\widetilde{x}, \widetilde{y}) = \widetilde{c}(\widetilde{y}, \widetilde{x})$ for all $\widetilde{x}, \widetilde{y}$.)

As in Example 1.4, we can define V to be the set of all diagonals of the N-gon, and Δ to be the collections of non-crossing diagonals. Then (1.1) defines $X(\Delta)$. We observe that

$$X(\widetilde{c}) \simeq X(\Delta) \cap \{p \mid u_x(p) = u_{\bar{x}}(p) \mid \forall x \in \Delta\},$$

where we write $\bar{x}$ for the 180-degree rotation of x.

It now follows $\Sigma(\widetilde{c}) = \Delta(\widetilde{c})$.

Problem 2.2. *For what compatibility degree functions c is it the case that $\Sigma(c) = \Delta(c)$?*

3. Examples from representation theory

Let A be a finite-dimensional algebra over a field k, with n isomorphism classes of simple modules, $S_1, \dots, S_n$. Let P_i be the projective cover of S_i, and I_i its injective hull. (A good introduction to the representation theory of finite-dimensional algebras is [**4**].)

Let V_A be the set of indecomposable A-modules, together with n additional objects, which we denote $P_i[1]$.

(There is a slightly more sophisticated framework in which we could work, which is a certain extension-closed subcategory of the homotopy category $K^b(\operatorname{proj} A)$. The category $K^b(\operatorname{proj} A)$ is triangulated, and in that context, $P_i[1]$ is the shift of P_i, but for simplicity we will use the simpler framework in which $P_i[1]$ is just a formal object, much as one adds n negative roots to the set of positive roots in order to account for the initial variables in a cluster algebra.)

To define the compatibility degree, we need another preliminary definition. Given an indecomposable A-module M, Let τM be the Auslander–Reiten translation of M. We also define $\tau(P_i[1]) = I_i$, and $\operatorname{Hom}(P_i[1], M) = 0$ for any module M.

Now define the compatibility degree of two objects of V_A by saying

$$c_A(X, Y) = \dim \operatorname{Hom}(X, \tau Y) + \dim \operatorname{Hom}(Y, \tau X).$$

$\Delta(c_A)$ is the abstract simplicial complex corresponding to the g-fan, as studied in [**5, 12, 1**].

We have the following theorem, which will be established in [**2**].

THEOREM 3.1. *Let A be a finite-dimensional algebra of finite representation type. Then $\Sigma(c_A) = \Delta(c_A)$.*

Note that in this case, if a module M has $\operatorname{Hom}(M, \tau M) \neq 0$, then the corresponding vertex does not appear in any face of $\Delta(c_A)$, but the corresponding variable still plays a role.

EXAMPLE 3.2. Fix a positive integer n. Let Q be the quiver with arrows α_i from i to $i+1$ and β_i from $i+1$ to i, for $1 \leq i \leq n-1$. Let $\mathbb{C}Q$ be the path algebra of this quiver, and let I be the ideal generated by $\alpha_i\beta_i$ and $\beta_i\alpha_i$ for $1 \leq i \leq n-1$. Let $A = \mathbb{C}Q/I$. It follows from combining the main result of [**11**] in the type A_n case with [**6**, Theorem 1.15] that $\Delta(c_A)$ is the type A_n Coxeter complex, which is dual to the permutohedron.

4. The geometrical problem

Beyond the combinatorial problem considered so far, we can also consider the following geometrical upgrade of the problem. Suppose that Δ can be realized as the boundary of a polytope, and let $P^\vee(\Delta)$ be the dual polytope.

PROBLEM 4.1.
1. *For what Δ, which can be realized as a simplicial polytope, is it the case that $X_{\geq 0}(\Delta)$ has a face-preserving diffeomorphism to $P^\vee(\Delta)$?*
2. *For what compatibility degree c, such that $\Delta(c)$ can be realized as a simplicial polytope, is it the case that $X_{\geq 0}(c)$ has a face-preserving diffeomorphism to $P^\vee(\Delta(c))$?*

A positive answer is shown in [**3**] for the c's that give rise to *cluster configuration spaces*. This includes Example 1.4 and Example 2.1. More precisely, the results from [**3**] cover exactly the examples in Section 3 which come from Q an orientation of a simply-laced Dynkin diagram, together with those that come from the natural extension of Section 3 to non-simply-laced Dynkin diagrams, via the theory of species. (For the theory of species, see, for example, [**7**].)

Currently no results are known beyond these cases.

5. Motivation

A system of equations equivalent to those coming from Example 1.4 were studied in 1969 by physicists Koba and Nielsen [**8, 9, 10**]. As in that example, let V be the set of diagonals of an N-gon, and Δ the collections of non-crossing diagonals.

Koba and Nielsen defined the "tree string amplitude" as an integral over $X_{\geq 0}(\Delta)$. The structure of the boundary of $X_{\geq 0}(\Delta)$ is important because of its implication for factorization properties of the integral.

This integral should be viewed as being associated to a disk; in perturbative quantum field theory one then expect higher corrections to the tree string amplitude associated to surfaces of higher genus.

As in Section 3, one can associate algebras to these surfaces, via the correspondence between triangulated surfaces and gentle algebras [**13**]. However, unlike in Section 3, these algebras are not of finite representation type, i.e., they have an infinite number of indecomposable modules.

Problem 5.1. *Formulate an analogue of (2.1) which extends the set-up of Section 3 to algebras having infinitely many indecomposable modules.*

This problem is intentionally somewhat vague. One possible approach would be to consider the variables u_x taking values in a ring of power series, in which the products (now infinite) in (2.1) have a chance to converge.

The goal is to define integrals providing higher loop corrections to the Koba–Nielsen tree string amplitude.

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