

COMPLEXITY AND ASYMPTOTICS OF STRUCTURE CONSTANTS

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“Thinking out of the black box”

ABSTRACT. Kostka, Littlewood-Richardson, Kronecker, and plethysm coefficients are fundamental quantities in algebraic combinatorics, yet many natural questions about them stay unanswered for more than 80 years. Kronecker and plethysm coefficients lack “nice formulas”, a notion that can be formalized using computational complexity theory. Beyond formulas and combinatorial interpretations, we can attempt to understand their asymptotic behavior in various regimes, and inequalities they could satisfy. Understanding these quantities has applications beyond combinatorics. On the one hand, the asymptotics of structure constants is closely related to understanding the [limit] behavior of vertex and tiling models in statistical mechanics. More recently, these structure constants have been involved in establishing computational complexity lower bounds and separation of complexity classes like VP vs VNP, the algebraic analogs of P vs NP in arithmetic complexity theory. Here we discuss the outstanding problems related to asymptotics, positivity, and complexity of structure constants focusing mostly on the Kronecker coefficients of the symmetric group and, less so, on the plethysm coefficients.

This expository paper is based on the talk presented at the Open Problems in Algebraic Combinatorics conference in May 2022.

1. INTRODUCTION

Algebraic Combinatorics studies symmetries via their manifestations in discrete objects, connecting with areas like Representation Theory, Statistical Mechanics, Computational Complexity Theory, and Algebraic Geometry. Many of its problems arise from the need for a quantitative and explicit understanding of algebraic phenomena like group representations and decompositions into irreducible representations, dimension formulae for modules, intersection numbers in geometry, etc.

In its origin lies the representation theory of the symmetric group S_n and general linear group GL_N . Their irreducible modules are indexed by integer partitions, and their bases can be described via standard and semi-standard Young tableaux (SYTs and SSYTs). As a next step, it is natural to understand how tensor products of irreducible representations decompose into irreducible components. In the case of GL_N the multiplicities of isotypic components are the Littlewood-Richardson (LR) coefficients. For S_n the multiplicities are the Kronecker coefficients. The composition of irreducible GL modules decomposes into irreducibles with multiplicities given by the plethysm coefficients.

While no compact formula for the LR coefficients exists, they can be understood via their combinatorial interpretation, the LR rule, which gives them in the form of certain skew SSYTs. The LR rule was formulated in 1934, but it took 40 years to prove formally. Inspired by the LR coefficients, Murnaghan defined the Kronecker coefficients in 1938 and observed that computing them is a highly nontrivial task. In late 70s and 80s the research turned towards obtaining positive formulas. In 2000 Stanley stated as a major open problem in Algebraic Combinatorics the problem of finding a nonnegative combinatorial interpretation for the Kronecker coefficients, as well as the plethysm coefficients. However, the progress has been minimal.

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 To appear in *Proceedings of the Open Problems in Algebraic Combinatorics* conference, May 2022.
<https://www.samuelhofkins.com/OPAC/opac.html>.

Besides presenting fundamental natural questions needing answers, the Kronecker and plethysm coefficients also play an important role in algebraic complexity theory, specifically Geometric Complexity Theory (GCT). Algebraic complexity theory studies the complexity of computing formal polynomials using arithmetic circuits, and some of its main problems concern the separation of complexity classes $VP \neq VNP$, the algebraic analog of the $P \neq NP$ Millennium problem. GCT approaches these problems by studying the symmetries and geometric properties of the universal polynomials (usually determinant and permanent), passing onto the coordinate rings of GL orbit closures of these polynomials. Complexity lower bounds and separation of classes then reduces to understanding the GL -irreducible representations of these rings and comparing their multiplicities. Such multiplicities are closely related to LR, Kronecker, and plethysm coefficients, and separating complexity classes can be achieved by proving certain inequalities for such multiplicities.

Another application of Algebraic Combinatorics is in Statistical Mechanics, Asymptotic Representation Theory and Random Matrix Theory. Some of the early connections date to the remarkable story of the longest increasing subsequence of permutations, which has the same [Tracy-Widom] distribution as the maximal eigenvalue of a random Hermitian matrix and the height of an SYT from the Plancherel measure. Symmetric polynomials, and particularly Schur functions, appear extensively in the study of random lozenge tilings, exclusion processes and vertex models. Plane partitions themselves are lozenge tilings with certain boundaries. Conversely, probabilistic and asymptotic methods have been used to understand algebro-combinatorial quantities. In the lack of compact formulas and combinatorial interpretations, understanding the growth behavior of structure constants is a natural next step. Beyond the interaction with Statistical Mechanics, understanding their growth is closely related to their application in GCT mentioned above.

Paper structure. Here we will briefly define the relevant structure constants, give some background, and phrase the open problems of three separate aspects: positivity, complexity, asymptotics. For more background on Complexity Theory and its specific connections with Algebraic Combinatorics we refer to the companion paper [Pan23] and references therein.

This paper presents the professional view of the author and is centered around problems on which she has worked on. It has no aspirations to be a comprehensive survey on the topics.

Acknowledgements. The author is grateful to her collaborators throughout the years and in particular Christian Ikenmeyer and Igor Pak for the extensive work on the subject of this paper. We would like to dedicate this to the memory of Christine Bessenrodt, whose deep and insightful work in the area has led a lot of the progress and with whom the author had the honor to collaborate briefly in 2020. We are grateful to the organizers of “Open Problems in Algebraic Combinatorics” for the inspiring conference and the opportunity to present these problems.

The author has been partially supported by the NSF.

2. BASIC OBJECTS AND NOTIONS

We will assume the reader is familiar with basic notions in Algebraic Combinatorics like integer partitions, Young tableaux and symmetric functions. For a concise recap of those see [Pan23]. For details on the combinatorial sides see [Sta99, Mac95] and for the representation theoretic aspects see [Sag01, Ful97].

2.1. Partitions and Young tableaux. We use standard notation from [Mac95] and [Sta99, §7] throughout the paper.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a *partition* of size $n := |\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 1$. We write $\lambda \vdash n$ for this partition, and $\mathcal{P} = \{\lambda\}$ for the set of all partitions. The length of λ is denoted $\ell(\lambda) := \ell$. Denote by $p(n)$ the number of partitions $\lambda \vdash n$. Let $\lambda + \mu$ denote the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$

Special partitions include the *rectangular shape* $(a^b) = (a, \dots, a)$, b times, the *hooks shape* $(k, 1^{n-k})$, the *two-row shape* $(n-k, k)$, and the *staircase shape* $\rho_\ell = (\ell, \ell-1, \dots, 1)$.

A *Young diagram* of shape λ is an arrangement of squares $(i, j) \subset \mathbb{N}^2$ with $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. Let $\lambda \vdash n$. A *semistandard Young tableau* T of shape λ and *weight* α is an arrangement of α_k many integers k in squares of λ , which weakly increase along rows and strictly increase down columns, i.e. $T(i, j) \leq T(i, j+1)$ and $T(i, j) < T(i+1, j)$. For example, $\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 2 & 3 & 5 & \\ \hline 4 & 5 & & & \\ \hline \end{array}$ is an SSYT of shape $\lambda = (5, 4, 2)$ and type $\alpha = (2, 3, 1, 3, 2)$. Denote by $\text{SSYT}(\lambda, \alpha)$ the set of such tableaux, and $K(\lambda, \alpha) = |\text{SSYT}(\lambda, \alpha)|$ the *Kostka number*. A *standard Young tableau* (SYT) of shape λ is an SSYT of type (1^n) , and we have $f^\lambda := K_{\lambda, 1^n}$, which can be computed by the hook-length formula (HLF) of [FRT54].

2.2. Representations of S_n and GL_N . The irreducible representations of the *symmetric group* S_n are the *Specht modules* \mathbb{S}_λ and are indexed by partitions $\lambda \vdash n$. A basis for \mathbb{S}_λ can be indexed by the SYTs. In particular

$$\dim \mathbb{S}_\lambda = f^\lambda.$$

We have that $\mathbb{S}_{(n)}$ is the trivial representation assigning to every w the value 1 and \mathbb{S}_{1^n} is the sign representation.

The *character* $\chi^\lambda(w)$ of \mathbb{S}_λ can be computed via the *Murnaghan-Nakayama rule*. Let w have type α , i.e. it decomposes into cycles of lengths $\alpha_1, \alpha_2, \dots, \alpha_k$. Then

$$\chi^\lambda(w) = \chi^\lambda(\alpha) = \sum_{T \in MN(\lambda; \alpha)} (-1)^{ht(T)},$$

where MN is the set of rim-hook tableaux of shape λ and type α , so that the entries are weakly increasing along rows and down columns, and all entries equal to i form a rim-hook shape (i.e. connected, no 2×2 boxes) of length α_i . The height of each rim-hook is one less than the number of rows it spans, and $ht(T)$ is the sum of all these heights. For examples

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 1 & 2 & 2 & 3 & 4 & \\ \hline 2 & 2 & 3 & 3 & 4 & \\ \hline \end{array}$$

is a Murnaghan-Nakayama tableau of shape $(6, 5, 5)$, type $(3, 5, 6, 2)$ and has height $ht(T) = 1 + 2 + 2 + 1 = 6$.

The irreducible polynomial representations of $GL_N(\mathbb{C})$ are the *Weyl modules* V_λ and are indexed by all partitions with $\ell(\lambda) \leq N$. Their characters are exactly the Schur functions $s_\lambda(x_1, \dots, x_N)$, where x_1, \dots, x_N are the eigenvalues of $g \in GL_N(\mathbb{C})$.

2.3. Symmetric functions. Let $\Lambda[\mathbf{x}]$ be the ring of *symmetric functions* $f(x_1, x_2, \dots)$, where the symmetry means that $f(\mathbf{x}) = f(\mathbf{x}_\sigma)$ for any permutation σ of the variables, and f is a formal power series. The ring Λ_n of homogeneous symmetric functions of degree n has several important bases: the *homogenous symmetric functions* h_λ , *elementary symmetric functions* e_λ , *monomial symmetric functions* m_λ , *power sum symmetric functions* p_λ and *Schur functions* s_λ . The Schur functions can be defined as the generating functions for SSYTs of shape λ

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu,$$

but also using *Weyl's determinantal formula*

$$s_\lambda(x_1, \dots, x_\ell) = \frac{\det[x_i^{\lambda_j + \ell - j}]_{i,j=1}^\ell}{\prod_{i < j} (x_i - x_j)}$$

and the *Jacobi-Trudi* identity

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{i,j=1}^{\ell(\lambda)}.$$

The *Cauchy identity*

$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\lambda}(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

has a remarkable combinatorial proof given by the RSK correspondence between pairs of same shape SSYT and matrices with nonnegative integer entries.

2.4. Computational Complexity. We refer to [Aar16, Wig19, BCS97] for details on Computational Complexity classes, and to [Pak22+, Pan23] and references therein for the connections with Algebraic Combinatorics.

A *decision problem* is a computational problem, for which the output is Yes or No. There are two major complexity classes P and NP, subject of the P vs NP Millennium problem. P is the class of decision problems, where given any input of size n (number of bits required to encode it), such that the answer can be obtained in *polynomial time* denoted by $poly(n)$, i.e. there is a fixed k and an algorithm taking $O(n^k)$ many steps (elementary operations). NP is the class of decision problems, where if the answer is Yes, then it can be verified in polynomial time, i.e. there is a poly-time computable witness. Naturally, $P \subset NP$ and it is widely believed that $P \neq NP$. The classes FP and #P are the counting analogues of P and NP. A *counting problem* is in #P if it is the number of accepting paths of an NP Turing machine. In practice, these are counting problems where the answers are exponentially large sums of 0-1 valued functions M , each of which can be computed in $O(n^k)$ time for a fixed k :

$$\sum_{b \in \{0,1\}^m} M(b),$$

where $m = poly(n)$. We set the classes $\text{GapP} = \{f - g \mid f, g \in \#P\}$ and $\text{GapP}_{\geq 0} = \{f - g \mid f, g \in \#P \text{ and } f - g \geq 0\}$.

The algebraic complexity classes VP and VNP were introduced by Valiant [V79a], as the algebraic analogues of P and NP (we refer to [Bür00a] for formal definitions and properties). They concern the computation of polynomials $f \in \mathbb{F}[x_1, \dots, x_N]$ of degree $poly(N)$ using arithmetic circuits where the inputs are constants of arbitrary size from the field \mathbb{F} and the formal variables x_1, \dots, x_n , the gates are $+$, \times , and the output is f . Then VP is the class of polynomials for which there is a fixed k and a circuit of size $O(n^k)$ computing the polynomial f . VNP is the class of polynomials f , such that there exist a k and $m = O(N^k)$ and a polynomial $g \in \text{VP}$ in $N + m$ variables, such that

$$f(x_1, \dots, x_N) = \sum_{b \in \{0,1\}^m} g(x_1, \dots, x_N, b_1, \dots, b_m).$$

The main conjecture is that $\text{VP} \neq \text{VNP}$ and is closely related to $P \neq NP$, see e.g. [Bür00b]. As Valiant showed, for every polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ there exists a K and a $K \times K$ matrix $A := A_0 + \sum_{i=1}^N A_i x_i$ with $A_j \in \mathbb{C}^{K \times K}$ such that $\det A = f$. The smallest such K is the *determinantal complexity* of f denoted $\text{dc}(f)$ and it is finite for every f . The VP_{ws} -universal polynomial is the determinant, in the sense that $f \in \text{VP}_{\text{ws}}$ iff $\text{dc}(f) = poly(n)$, where $poly(n)$ denotes any fixed degree polynomial in n . Here VP_{ws} is the class of polynomials with poly-sized Algebraic Branching Programs (a model between “formula” and “circuit”). The class VP_{ws} has been the main subject of study and is often interchangeably used for VP. The classical universal VNP-complete polynomial is the permanent

$$\text{per}_m[X_{ij}]_{i,j=1}^m = \sum_{\sigma \in S_m} \prod_{i=1}^m X_{i\sigma(i)}.$$

Thus, separating VP_{ws} from VNP can be done by showing that $\text{dc}(\text{per}_m) \gg poly(m)$.

The *Geometric Complexity Theory* (GCT) program of Mulmuley and Sohoni [MS01, MS08] compares the determinant and permanent by comparing their orbit closures under the action of GL_{n^2} on the variables (X_{11}, \dots, X_{nn}) . If $\text{dc}(\text{per}_m) \leq n$, then $\overline{GL_{n^2}\text{per}_m} \subset \overline{GL_{n^2}\text{det}_n}$, and passing to the coordinate ring, this induces a GL_{n^2} -equivariant surjection

$$(2.1) \quad \mathbb{C}[\overline{GL_{n^2}\text{det}_n}] \rightarrow \mathbb{C}[\overline{GL_{n^2}X_{11}^{n-m}\text{per}_m}]$$

To show a lower bound for $\text{dc}(\text{per}_m)$, i.e. $\text{dc}(\text{per}_m) > n$, then we need to show that the above surjection does not happen. Decomposing these modules into irreducibles for any given degree d , we have

$$(2.2) \quad \mathbb{C}[\overline{GL_{n^2}\text{det}_n}]_d \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus \delta_{\lambda,d,n}}, \quad \text{and} \quad \mathbb{C}[\overline{GL_{n^2}\text{per}_m}]_d \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus \gamma_{\lambda,d,n,m}}.$$

Thus, to show that 2.1 is not possible, we can find an irreducible GL_{n^2} module V_{λ} , such that $\gamma_{\lambda,d,n,m} > \delta_{\lambda,d,n}$ for some d . Such λ 's are called *multiplicity obstructions* and γ and δ are closely related to plethysm and Kronecker coefficients. When $\delta_{\lambda,d,n} = 0$ also, then λ is an *occurrence obstruction*. It was shown in [BIP19] that there are no occurrence obstructions for $n > m^{25}$, so we need to find multiplicity obstructions.

This is one of the main applications of Kronecker and plethysm coefficients and motivates our work. In particular, we have

$$\delta_{\lambda,d,n} \leq g(\lambda, n^d, n^d) \quad \text{and} \quad \gamma_{\lambda,d,n,m} \leq a_{\lambda}(d[n]).$$

See §3 for the definitions and [Pan23] for a brief overview of these connections.

3. MULTIPLICITIES AND STRUCTURE CONSTANTS

Once the irreducible representations have been sufficiently understood, it is natural to consider what other representations can be formed by them and how such representations decompose into irreducibles.

In the case of $GL_N(\mathbb{C})$ these coefficients are the *Littlewood–Richardson coefficients* (LR) $c_{\mu\nu}^{\lambda}$ defined as the multiplicity of V_{λ} in $V_{\mu} \otimes V_{\nu}$, so

$$V_{\mu} \otimes V_{\nu} = \bigoplus_{\lambda} V_{\lambda}^{\oplus c_{\mu\nu}^{\lambda}}.$$

Via their characters, they can be equivalently defined as

$$s_{\mu}s_{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}, \quad \text{and} \quad s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

While no nice product formula exists for their computation, they have a *combinatorial interpretation*, the so called *Littlewood–Richardson rule*, see [Sta99, App. 1].

Theorem 3.1 (Littlewood–Richardson rule). *The Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda}$ is equal to the number of skew SSYT T of shape λ/μ and type ν , whose reading word is a ballot sequence.*

A ballot sequence means that in every prefix of the sequence the number of 1's \geq the number of 2's \geq number of 3's etc. For example, we have $c_{(3,1),(4,3,2)}^{(6,4,3)} = 2$ as there are two LR tableaux (SSYT

of shape $(6, 4, 3)/(3, 1)$ and type $(4, 3, 2)$ whose reading words are ballot sequences)

1	1	1	1
2	1	2	2
2	3	3	

with reading word 111221332 and

1	1	1	1
2	2	2	2
1	3	3	

 with reading word 111222331.

The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ of the symmetric group are the corresponding structure constants for the ring of S_n -irreducibles. Namely, S_n acts diagonally on the tensor product of two Specht modules and the corresponding module factors into irreducibles with multiplicities $g(\lambda, \mu, \nu)$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \bigoplus_{\nu} \mathbb{S}_\nu^{\oplus g(\lambda, \mu, \nu)}.$$

In terms of characters we can write them as

$$(3.1) \quad g(\lambda, \mu, \nu) = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) \chi^\mu(w) \chi^\nu(w).$$

The last formula shows that they are symmetric upon interchanging the underlying partitions $g(\lambda, \mu, \nu) = g(\mu, \nu, \lambda) = \dots$ which motivates us to use such symmetric notation.

The *Kronecker product* $*$ on Λ is defined on the Schur basis by

$$s_\lambda * s_\mu = \sum_{\nu} g(\lambda, \mu, \nu) s_\nu,$$

and extended by linearity.

Inspired by the Littlewood-Richardson coefficients, Murnaghan defined them in 1938. In fact, he showed that, see [Mur56],

Theorem 3.2 (Murnaghan). *For every λ, μ, ν , such that $|\lambda| = |\mu| + |\nu|$ we have that*

$$c_{\mu\nu}^\lambda = g((n - |\lambda|, \lambda), (n - |\mu|, \mu), (n - |\nu|, \nu)),$$

for sufficiently large n .

In particular, one can see that $n = 2|\lambda| + 1$ would work. Note that even when the sizes of the partitions do not add up, the RHS stabilizes as $n \rightarrow \infty$ to the *reduced Kronecker coefficients*, see 7.

Thanks to the Schur-Weyl duality, they can also be interpreted via Schur functions as

$$s_\lambda[\mathbf{x}\mathbf{y}] = \sum_{\mu, \nu} g(\lambda, \mu, \nu) s_\mu(\mathbf{x}) s_\nu(\mathbf{y}),$$

where $\mathbf{x}\mathbf{y} = (x_1y_1, x_1y_2, \dots, x_2y_1, \dots)$ is the vector of all pairwise products. In terms of GL representations they give us the dimension of the invariant space

$$g(\lambda, \mu, \nu) = \dim(V_\mu \otimes V_\nu \otimes V_\lambda^*)^{GL_N \times GL_M},$$

where V_μ is considered as a GL_N -module, and V_ν as a GL_M -module and V_λ^* is a dual GL_{NM} -module. In other words, they are dimensions of highest weight vector spaces, and using this interpretation for them as dimensions of highest weight spaces one can show the following, see [CHM07]

Theorem 3.3 (Semigroup property). *Let $(\alpha^1, \beta^1, \gamma^1)$ and $(\alpha^2, \beta^2, \gamma^2)$ be two partition triples, such that $|\alpha^1| = |\beta^1| = |\gamma^1|$ and $|\alpha^2| = |\beta^2| = |\gamma^2|$. Suppose that $g(\alpha^1, \beta^1, \gamma^1) > 0$ and $g(\alpha^2, \beta^2, \gamma^2) > 0$. Then*

$$g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max\{g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)\}.$$

From the original S_n characters definition we see that

$$g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu)$$

since $\chi^{\lambda'}(w) = \chi^\lambda(w) \operatorname{sgn}(w)$. In particular, we have $g(\lambda, \mu, (n)) = \delta_{\lambda, \mu}$ for all λ and $g(\lambda, \mu, 1^n) = \delta_{\lambda, \mu'}$.

Example 3.4. *By the above observation we have that*

$$h_k[\mathbf{x}\mathbf{y}] = s_k[\mathbf{x}\mathbf{y}] = \sum_{\lambda \vdash k} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}).$$

Using the Jacobi-Trudi identity we can write

$$\begin{aligned} s_{2,1}[\mathbf{xy}] &= h_2[\mathbf{xy}]h_1[\mathbf{xy}] - h_3[\mathbf{xy}] = (s_2(\mathbf{x})s_2(\mathbf{y}) + s_{1,1}(\mathbf{x})s_{1,1}(\mathbf{y}))s_1(\mathbf{x})s_1(\mathbf{y}) \\ &\quad - s_3(\mathbf{x})s_3(\mathbf{y}) - s_{2,1}(\mathbf{x})s_{2,1}(\mathbf{y}) - s_{1,1,1}(\mathbf{x})s_{1,1,1}(\mathbf{y}) \\ &= s_{2,1}(\mathbf{x})s_{2,1}(\mathbf{y}) + s_{2,1}(\mathbf{x})s_3(\mathbf{y}) + s_3(\mathbf{x})s_{2,1}(\mathbf{y}) + s_{1,1,1}(\mathbf{x})s_{2,1}(\mathbf{y}) + s_{2,1}(\mathbf{x})s_{1,1,1}(\mathbf{y}). \end{aligned}$$

So we see that $g((2,1), (2,1), (2,1)) = 1$.

The *plethysm coefficients* $a_{\mu,\nu}^\lambda$ are multiplicities of an irreducible GL representation in the composition of two GL representations. Namely, let $\rho^\mu : GL_N \rightarrow GL_M$ be one irreducible, and $\rho^\nu : GL_M \rightarrow GL_K$ be another. Then $\rho^\nu \circ \rho^\mu : GL_N \rightarrow GL_K$ is another representation of GL_N which has a character $s_\nu[s_\mu]$, which decomposes into irreducibles as

$$s_\nu[s_\mu] = \sum_{\lambda} a_{\mu,\nu}^\lambda s_\lambda.$$

Here the *plethystic* notation $f[g]$ means the evaluation of f over the monomials of g as variables: if $g = \mathbf{x}^{\alpha^1} + \mathbf{x}^{\alpha^2} + \dots$, then $f[g] = f(\mathbf{x}^{\alpha^1}, \mathbf{x}^{\alpha^2}, \dots)$.

Example 3.5. We have that

$$\begin{aligned} s_{(2)}[s_{(1^2)}] &= h_3[e_2] = h_2(x_1x_2, x_1x_3, \dots) = x_1^2x_2^2 + x_1^2x_2x_3 + 3x_1x_2x_3x_4 + \dots \\ &= s_{2,2}(x_1, x_2, x_3, \dots) + s_{1,1,1,1}(x_1, x_2, x_3, \dots), \end{aligned}$$

so $a_{(2),(1,1)}^{(2,2)} = 1$ and $a_{(2),(1,1)}^{(3,1)} = 0$.

We will be particularly interested when $\mu = (d)$ or (1^d) which are the d th symmetric power Sym^d and the d th wedge power Λ^d , and $\nu = (n)$. We denote this plethysm coefficient by $a_\lambda(d[n]) := a_{(d),(n)}^\lambda$ and

$$h_d[h_n] = \sum_{\lambda} a_\lambda(d[n])s_\lambda.$$

These coefficients are central to the GCT application and deserve special attention.

4. COMBINATORIAL INTERPRETATIONS

“Combinatorial interpretation”, albeit never formally defined in the literature, is assumed to mean a family of nice discrete objects whose cardinality gives the desired quantity. In practice, for quantities with combinatorial interpretation, computing them is in $\#\mathbf{P}$. On the other hand, being in $\#\mathbf{P}$ does not necessarily give a “nice” combinatorial interpretation, but is certainly a starting point. All problems considered here are in \mathbf{GapP} , which makes $\#\mathbf{P}$ the natural class to consider in relation to “manifestly nonnegative” formulas. If such a problem is not in $\#\mathbf{P}$, then there could not be a reasonable nonnegative combinatorial interpretation within the given context. This paradigm is discussed at length in [IP22, Pak22+, Pan23].

Following the discovery of the Littlewood-Richardson rule in 1934, Murnaghan [Mur38] defined the Kronecker coefficients of S_n and computed various special cases, but quickly observed that this is no easy task. Interest in nonnegative combinatorial interpretation of these coefficients reemerged in the 80s, see [Las79, GR85], and was stated as Problem 10 in Stanley’s list¹ of “Open Problems in Algebraic Combinatorics” [Sta00].

Open Problem 4.1 (Stanley). *Find a combinatorial interpretation of $g(\lambda, \mu, \nu)$, thereby combinatorially proving that they are nonnegative.*

¹See this for the original list and updates on the problems <https://mathoverflow.net/questions/349406/>

Over the years, there has been very little progress on the question. In 1989 Remmel determined $g(\lambda, \mu, \nu)$ when two of the partitions are hooks. In 1994 Remmel and Whitehead [RW94] determined $g(\lambda, \mu, \nu)$ when two of the partitions are two-rows, i.e. $\ell(\lambda), \ell(\mu) \leq 2$. This case was subsequently studied also in [BMS15]. In 2006 Ballantine and Orellana [BO05] determined a rule for $g(\lambda, \mu, \nu)$ when one partition is a two-row, e.g. $\mu = (n - k, k)$, and the first row of one of the others is large, namely $\lambda_1 \geq 2k - 1$. The most general rule was determined by Blasiak [B13] in 2012 when one partition is a hook, and this was later simplified by Blasiak and Liu [BL18], and further in [Liu17]. Other very special cases have been computed for multiplicity free Kronecker products by Bessenrodt-Bowman [BB17], Kroneckers corresponding to pyramids by Ikenmeyer-Mulmuley-Walter [IMW17], $g(m^k, m^k, (mk - n, n))$ as counting labeled trees by Pak-Panova see Remark 4.4, near rectangular partitions by Tewari in [T15], etc.

Similar questions pertain to the plethysm coefficients. The following problem is number 9 in Stanley's list [Sta00].

Open Problem 4.2 (Stanley). *Find a combinatorial interpretation for the plethysm coefficients $a_{\mu, \nu}^\lambda$.*

A detailed survey on the partial results and methods can be found in [COSSZ22].

Even the simple case for $a_\lambda(d[n]) = \langle s_\lambda, h_d[h_n] \rangle$ is not known in general unless $n = 2$. While, there is no direct connection between the Kronecker and plethysm coefficients, the case when $\ell(\lambda) = 2$ coincide, see [PP14], and actually give a combinatorial interpretation as explained below.

Proposition 4.3. *We have that $g(\lambda, n^d, n^d) = a_\lambda(d[n]) = p_k(n, d) - p_{k-1}(n, d)$ for $\lambda = (nd - k, k) \vdash nd$. Here $p_r(a, b) = \#\{\mu \vdash r : \mu_1 \leq a, \ell(\mu) \leq b\}$ are the partitions of r which fit inside the $a \times b$ rectangle.*

Remark 4.4. In particular, these are the coefficients in the q -binomials

$$\sum_r p_r(a, b) q^r = \binom{a+b}{a}_q := \prod_{i=1}^a \frac{(1 - q^{i+b})}{1 - q^i}.$$

The combinatorial proof by Kathy O'Hara of the unimodality of the $p_r(a, b)$ is what gives the combinatorial interpretation for the particular Kronecker and plethysm coefficients as counting certain labeled trees, see [Pan15, p. 9].

To formalize these problems, we formulate them within the Computational Complexity framework. To set the stage with an example, we start with the Littlewood-Richardson coefficients.

ComputeLR:

Input: λ, μ, ν

Output: Value of $c_{\mu\nu}^\lambda$.

Here and below, the input can be encoded in *unary* or *binary*, and this could affect the complexity. By unary we mean that the input for each partition is written as $(\underbrace{11\dots 1}_{\lambda_1}, \underbrace{1\dots 1}_{\lambda_2}, \dots)$, so the

input size is $O(|\lambda|) = O(n)$. In binary, we encode each part λ_i in base 2, so the input size is $\log_2(\lambda_1) + \log_2(\lambda_2) + \dots = O(\ell(\lambda) \log_2(\lambda_1))$.

By the Littlewood-Richardson rule we have that when the input is unary, $\text{ComputeLR} \in \#\text{P}$, as our witnesses are the skew λ/μ tableaux, which are at most $n^n = O(2^{n^2})$ -many, and each can be checked to be an LR tableaux in time $O(n^2)$. The LR conditions can be translated into $O(\ell(\lambda)^2)$ many linear inequalities, which comprise the LR polytope. This description gives us that if the input is in binary we still have $\text{ComputeLR} \in \#\text{P}$.

Using a reduction to Knapsack for binary input, it is not difficult to see that

Theorem 4.5 (Narayanan [Nar06]). *When the input λ, μ, ν is in binary, `ComputeLR` is $\#P$ -complete.*

When the input is in unary, we do not know whether the problem is still that hard.

Conjecture 4.6 (Pak-Panova 2020). *When the input is in unary we have that `ComputeLR` is $\#P$ -complete.*

None of the above has been achieved for the Kronecker and plethysm coefficients, however, due to the lack of any positive combinatorial formula. Mulmuley also conjectured that computing the Kronecker coefficients would be in $\#P$, mimicking the Littlewood-Richardson coefficients.

`ComputeKron`:

Input: λ, μ, ν

Output: Value of $g(\lambda, \mu, \nu)$.

Conjecture 4.7 (Pak). *`COMPUTEKRON` is not in $\#P$ under reasonable complexity theoretic assumptions like PH not collapsing.*

If the above is proven, that would make any solution to Open Problem 4.1 as unlikely as the polynomial hierarchy collapsing. Any reasonable combinatorial interpretation such as counting certain objects would show that the problem is in $\#P$, since the objects would likely be verifiable in polynomial time. The problem stands for both unary and binary input.

Note that `ComputeKron` \in GapP ([BI08]) as it is easy to write an alternating sum for its computation, for example using contingency arrays, see [PP17b].

The author's experience with Kronecker coefficients suggests that some particular families would be as hard as the general problem.

Conjecture 4.8 (Panova). *`ComputeKron` is in $\#P$ when $\ell(\lambda) = 2$ if and only if `ComputeKron` is in $\#P$. Likewise, `ComputeKron` is in $\#P$ for $\mu = \nu = (n^d)$ and $\lambda \vdash nd$ as the input if and only if `ComputeKron` is in $\#P$.*

Of course, the backward direction is clear, if the general problem is in $\#P$, then also the special subproblems are in $\#P$.

The plethysm coefficients can also be studied from this perspective.

`ComputePleth`:

Input: λ, μ, ν

Output: Value of $a_{\mu, \nu}^\lambda$.

Open Problem 4.9. *Determine whether `ComputePleth` \in $\#P$.*

Using symmetric function identities, it is not hard to find an alternating formula for the plethysms and show that they are also in GapP , see [FI20]. They also show that deciding positivity is NP-hard and thus the computational problem is $\#P$ -hard. `ComputePleth` may not be in $\#P$ in the general case, but also when μ, ν are single row partitions. The coefficient $a_\lambda(d[n])$ in this case has special significance in GCT, see [Pan23].

Underlying all the representation theoretic multiplicities mentioned above are the characters of the symmetric group. For example, equation (3.1) expresses the Kronecker coefficients via characters, and the other structure constants can also be expressed in similar ways. This motivates us to understand them also computationally. As we shall see, they also provide a proof of concept for the idea that certain positive integral algebro-combinatorial quantities may indeed not have a combinatorial interpretation.

The characters satisfy some particularly nice identities coming from the orthogonality of the rows and columns of the character table in S_n . We have that

$$(4.1) \quad \sum_{\lambda \vdash n} \chi^\lambda(w)^2 = \prod_i i^{c_i} c_i!,$$

where c_i = number of cycles of length i in $w \in S_n$. When $w = \text{id}$, we have that $\chi^\lambda(\text{id}) = f^\lambda$, the number of SYTs and the identity is proven using the beautiful RSK bijection. The first step in this proof is to identify $(f^\lambda)^2$ as the number of pairs of SYTs of the same shape.

Could anything like that be done for equation (4.1)? The first step would be to understand what objects $\chi^\lambda(w)^2$ counts, does it have any positive combinatorial interpretation? We formulate it again using the CC paradigm as

ComputeCharSq:

Input: $\lambda, \alpha \vdash n$, unary.

Output: the integer $\chi^\lambda(\alpha)^2$.

Theorem 4.10 ([IPP22]). *ComputeCharSq* $\notin \#P$ unless $PH = \Sigma_2^P$.

The last condition says ‘‘polynomial hierarchy collapses to the second level’’, which is almost as unlikely as $P = NP$. The widely believe complexity theoretic assumptions are that $P \neq NP$ and that PH does not collapse. This problem shows that some natural questions in Algebraic Combinatorics are indeed not in $\#P$.

The proof of this Theorem follows from another complexity theoretic result we show, namely that deciding if $\chi^\lambda(\alpha) \neq 0$ is PH -hard as well, see end of §5.

5. POSITIVITY PROBLEMS

Motivated by other developments, further questions about the Kronecker coefficients have appeared. Following their work in [HSTZ13] on the square of the Steinberg character for finite groups of Lie type, the authors conjectured that a similar irreducible character should exist for the symmetric group and formulated the following.

Conjecture 5.1 (Tensor square conjecture). *For every $n \geq 9$ there exists a symmetric partition $\lambda \vdash n$, such that $\mathbb{S}_\lambda \otimes \mathbb{S}_\lambda$ contains every irreducible S_n module. In other words $g(\lambda, \lambda, \mu) > 0$ for all $\mu \vdash n$.*

Following this, Saxl conjectured a specific partition, namely the staircase, which would satisfy that.

Conjecture 5.2 (Saxl, see [PPV16]). *Let $\delta_k = (k, k-1, \dots, 1)$ be the staircase partition. Then $g(\delta_k, \delta_k, \mu) > 0$ for all k and $\mu \vdash \binom{k+1}{2}$.*

This conjecture has received a lot of attention partially due to its concreteness.

Positivity results were proved using a combination of three methods – the semigroup property constructing recursively positive triples from building blocks, explicit highest weight constructions using the techniques in [Ful00], and an unusual comparison with characters, which was originally stated in by Bessenrodt and Behns [BB04] to show that $g(\lambda, \lambda, \lambda) > 0$ for $\lambda = \lambda'$, later generalized in [PPV16], and in its final form in [PP17a].

Theorem 5.3 ([PP17a]). *Let $\lambda, \mu \vdash n$ and $\lambda = \lambda'$. Let $\hat{\lambda} = (2\lambda_1 - 1, 2\lambda_2 - 3, 3\lambda_3 - 5, \dots)$ be the principal hooks of λ . Then*

$$g(\lambda, \lambda, \mu) \geq |\chi^\mu(\hat{\lambda})|.$$

The technique of explicit highest weight vectors construction led to the following more general result.

Theorem 5.4 ([I15]). *Let $\mu \vdash \binom{k+1}{2}$ be a partition comparable to ρ_k in the dominance order, i.e. $\mu \succ \rho_k$ or $\rho_k \succ \mu$. Then $g(\rho_k, \rho_k, \mu) > 0$.*

Using this result and generalizations, and clever combinations of the semigroup property 3.3, Luo and Sellke [LS17] proved that $\chi^{\rho_k} \otimes \chi^{\rho_k} \otimes \chi^{\rho_k} \otimes \chi^{\rho_k}$ contains every χ^μ , and that $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains “most” irreducible representations. In [B18], Bessenrodt proved that all double hooks, i.e. partitions with Durfee size 2, are contained in $\chi^{\rho_k} \otimes \chi^{\rho_k}$, and Li generalized to triple hooks [L21]. Harman and Ryba [HR22] showed that $\chi^{\rho_k} \otimes \chi^{\rho_k} \otimes \chi^{\rho_m}$ contains every irreducible representation.

The semigroup property is a powerful tool to create positive Kronecker triples by induction out of building blocks. However, creating a particular partition, like the staircase or a rectangle, out of building blocks is very restrictive and quickly encounters number theoretic issues. A rectangle can be “cut” only into other rectangles. This prompts us to enlarge the class of positive triples. In 2020, with Christine Bessenrodt we generalized Conjecture 5.1 as follows (and checked for partitions up to size 20).

Conjecture 5.5 (Bessenrodt-Panova 2020). *For every n there exists a $k(n)$, such that for every $\lambda \vdash n$ with $\lambda = \lambda'$ and $d(\lambda) > k_n$ which is not the square partition, we have $g(\lambda, \lambda, \mu) > 0$ for all $\mu \vdash n$.*

Here $d(\lambda) = \max\{i : \lambda_i \geq i\}$ is the Durfee square size of the partition. Partial progress on that conjecture will appear in the work of Chenchen Zhao.

Another positivity result, motivated by applications in GCT, was obtained in [IP17] as

Lemma 5.6 ([IP17, Thm 1.10]). *Let $X := \{1, 1^2, 1^4, 1^6, 21, 31\}$, and let partition $\nu \notin X$. Denote $\ell := \max\{\ell(\nu) + 1, 9\}$, and suppose $r > 3\ell^{3/2}$, $s \geq 3\ell^2$, and $|\nu| \leq rs/6$. Let $\nu[N] := (N - |\nu|, \nu_1, \nu_2, \dots)$. Then $g(s^r, s^r, \nu[rs]) > 0$.*

The tensor square conjectures raise the question on simply determining when $g(\lambda, \mu, \nu) > 0$. It is a consequence of representation theory that when $n > 2$ for every $\mu \vdash n$ there is a $\lambda \vdash n$, such that $g(\lambda, \lambda, \mu) > 0$, see [Sta99, Ex. 7.82] but even that has no combinatorial proof.

To understand why such problems are difficult, we resort to the computational complexity framework. For Kostka numbers, determining if $K_{\lambda, \mu} > 0$ is easy, it just requires us checking that $\lambda \succ \mu$ in the dominance order. For Littlewood-Richardson that’s a lot less trivial despite the combinatorial rule.

LRPos:

Input: λ, μ, ν
Output: Is $c_{\mu\nu}^\lambda > 0$?

The proof of the Saturation Conjecture by Knutson and Tao showed that an LR coefficient is nonzero if and only if the corresponding hive polytope is nonempty. This polytope is a refinement of the Gelfand-Tsetlin polytope, and is defined by $O(\ell(\lambda)^2)$ many inequalities. Showing that the polytope is nonempty is thus a linear programming problem, which can be solved in polynomial time, see [MNS12].

Theorem 5.7. *We have that LRPos $\in P$ when the input is in binary (and unary).*

KronPos:

Input: λ, μ, ν
Output: Is $g(\lambda, \mu, \nu) > 0$?

In the early stages of GCT Mulmuley conjectured [MS08] that they would behave like the Littlewood-Richardson, so KronPos $\in P$, which was recently disproved.

Theorem 5.8 ([IMW17]). *When the input λ, μ, ν is in unary, KronPos is NP-hard.*

The proof uses the fact that in certain cases $g(\lambda, \mu, \nu)$ is equal to the number of pyramids with marginals λ, μ, ν and deciding if there is such a pyramid is NP-complete. However, the problem is not yet in NP: we do not have, for every λ, μ, ν , polynomial witnesses showing that $g(\lambda, \mu, \nu) > 0$ when this is the case. This is yet another open problem.

Open Problem 5.9. *Determine whether when the input λ, μ, ν is in unary, $KronPos$ is in NP.*

Needless to say, the problem would be [computationally] harder when the input is in binary.

Positivity of plethysm coefficients is even less understood. Plethysm coefficients $a_\lambda(d[n])$ are important in GCT and positivity of $a_\lambda(d[n])$ for partitions λ with long first rows was established in [BIP19]. In general, we do not expect the problem to be any easier.

PlethPos:

Input: λ, μ, ν

Output: Is $a_{\mu, \nu}^\lambda > 0$?

In [FI20], Fischer and Ikenmeyer showed that **PlethPos** is NP-hard. As there is no combinatorial interpretation and no nice positivity criteria in general we cannot say whether the problem is in NP.

Open Problem 5.10. *Determine whether $PlethPos$ is in NP.*

There is one major conjecture on plethysm coefficients.

Conjecture 5.11 (Foulkes). *Let $d > n$, then*

$$a_\lambda(d[n]) \geq a_\lambda(n[d])$$

for all $\lambda \vdash nd$.

This conjecture is related to the Alon-Tarsi conjecture, and has appeared in GCT-related research. In [DIP19] we proved it for some families of 3-row partitions.

As most of these quantities can be computed via symmetric group characters, it is natural to ask how easy it is to decide if a character is 0 or not.

CharVanish:

Input: $\lambda, \alpha \vdash n$, unary.

Output: Is $\chi^\lambda(\alpha) = 0$?

Theorem 5.12 ([IPP22]). *We have that $CharVanish$ is $C=P$ -complete under many-to-one reductions.*

The class $C=P$ is defined formally as $[GapP = 0]$, the class of deciding when two $\#P$ functions have equal values, and this is a class believed to be strictly larger than $coNP$ and NP . Thus we cannot expect that deciding $\chi = 0$ would be in $coNP$ and $\chi \neq 0$ in NP . This also leads to the proof of Proposition 5.3. See [Pan23] for a discussion on that topic and sketch of the proof.

6. ASYMPTOTIC ALGEBRAIC COMBINATORICS

Asymptotic Algebraic Combinatorics is a natural extension of Algebraic Combinatorics inspired by Probability and Asymptotic Representation Theory. Instead of searching for exact formulas, bijections, combinatorial interpretations, we are looking to understand quantities approximately. This is, indeed, what we could hope for with the given structure constants in the absence of nice formulas.

As mentioned in the Introduction, such problems have natural interplay with Statistical Mechanics and Integrable Probability. They would also be important in the search of multiplicity obstructions in GCT, see [Pan23].

6.1. Maximal multiplicities. The first step in such asymptotic analysis begins with estimating the range of possible values. In [Sta99, Sta20], Stanley observed that

$$\begin{aligned} \max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} g(\lambda, \mu, \nu) &= \sqrt{n!} e^{-O(\sqrt{n})}, \\ \max_{0 \leq k \leq n} \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c_{\mu, \nu}^{\lambda} &= 2^{n/2 - O(\sqrt{n})}. \end{aligned}$$

and asked

Open Problem 6.1 (Stanley). *Fix n . For which λ, μ, ν with $\lambda \vdash n$, is $c_{\mu\nu}^{\lambda}$ asymptotically maximal? For which λ, μ, ν is $g(\lambda, \mu, \nu)$ asymptotically maximal?*

Using various symmetric function and representation theoretic identities involving sums of these coefficients, it was observed in [PPY19a] that such maximal partitions λ must also have asymptotically maximal dimensions f^{λ} . By the pioneering work of Vershik-Kerov [VK] and Loggan-Shepp [LoS], we know that such partitions approach certain explicit curve, which we refer to as the VKLS shape and VKLS partitions, and for all those we have $f^{\lambda} = \sqrt{n!} e^{-O(\sqrt{n})}$.

Theorem 6.2 ([PPY19a]). *Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ be three partition sequences, such that*

$$(*) \quad g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) = \sqrt{n!} e^{-O(\sqrt{n})}.$$

Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ have VKLS shape (i.e. are asymptotically maximal). Conversely, for every two VKLS sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash n\}$, there exists a VKLS partition sequence $\{\nu^{(n)} \vdash n\}$, s.t. () holds.*

This statement shows “existence”, the problem here is to exhibit a particular explicit family of partitions for which this asymptotics holds.

Open Problem 6.3. *Determine a family of partitions $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)} \vdash n$, such that*

$$g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) = \sqrt{n!} e^{-O(\sqrt{n})}$$

as $n \rightarrow \infty$.

We believe that for the staircase partitions $\rho_k = (k, k-1, \dots, 1)$ the Kronecker coefficient $g(\rho_k, \rho_k, \rho_k)$ grows superexponentially, and this is one specific candidate which could achieve large, but not maximal, values. However, no exponential lower bounds are known in this case.

Similar analysis can be done for the Littlewood-Richardson coefficients. Again, the proofs can only derive “existence”, but not exhibit any specific families.

Theorem 6.4 ([PPY19a]). *Fix $0 < \theta < 1$ and let $k_n := \lfloor \theta n \rfloor$. Then:*

1. for every VKLS partition sequence $\{\lambda^{(n)} \vdash n\}$, there exist VKLS partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$(**) \quad c_{\mu^{(n)}, \nu^{(n)}}^{\lambda^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})},$$

*2. for all VKLS partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, there exists a VKLS partition sequence $\{\lambda^{(n)} \vdash n\}$, s.t. (**) holds,*

3. for all VKLS partition sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash k_n\}$, there exists a partition sequence $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$f^{\nu^{(n)}} = \sqrt{n!} e^{-O(n^{2/3} \log n)} \quad \text{and} \quad c_{\mu^{(n)}, \nu^{(n)}}^{\lambda^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(n^{2/3} \log n)}.$$

We again ask when such maxima are achieved.

Open Problem 6.5. Let $\theta \in (0, 1)$ be some fixed number and let $k_n := \lfloor \theta n \rfloor$. Find families of partitions $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, for which

$$c_{\mu^{(n)}, \nu^{(n)}}^{\lambda^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})},$$

While asymptotics of Kronecker coefficients have barely been studied, Schur functions asymptotics in various regimes has been central to the field of Integrable Probability. It is closely related to the Harish-Chandra-Itzykson-Zuber integrals in random matrix theory. A step further takes us to Kostka and Littlewood-Richardson coefficients, which can also be estimated using the so-called elliptic integrals, see [BGH]. There, the problem of asymptotics is translated into an implicit variational analysis, and it still does not give explicit concrete answers.

There are currently no nontrivial asymptotic results of this kind for plethysm coefficients.

Open Problem 6.6. Determine the maximal asymptotic value of the plethysm coefficients

$$\max_{\lambda \vdash n^2} a_\lambda(n[n]).$$

The only nontrivial bounds come from their relation to rectangular Kronecker coefficients, and in particular when λ is a two-row partition, see § 6.3. We expect that the maxima should be at least exponential. Another source of nontrivial lower bounds can be found in [FI20] where plethysms are compared to pyramids. Similar comparison was used to obtain a family of large Kronecker coefficients in [PP20a, PP23].

6.2. Bounded rows and diagonals. In the “maximal” regime, the partitions involved were close to the VKLS shape which is symmetric and has length $O(\sqrt{n})$ for partitions of n . The asymptotics changes significantly depending on the regimes of convergence for the partitions involved.

When the lengths of the partitions are bounded, formulas coming from symmetric functions and relating them to contingency tables become useful. We have the following upper bound.

Theorem 6.7 ([PP20a]). Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:

$$g(\lambda, \mu, \nu) \leq \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}.$$

In particular, when $\lambda = \mu = \nu = ((\ell^2)^\ell) \vdash \ell^3 = n$, we have $g(\lambda, \mu, \nu) \leq 4^n$. Yet, a similar lower bound for that case is beyond reach.

Further analysis using Schur function techniques also gives bounds when the Durfee size (diagonal) of a partition is bounded, even if the length grows. We have the following, see [PP23].

Theorem 6.8 ([PP23]). Let $n, k \geq 1$, and let $\lambda, \mu, \nu \vdash n$, such that $d(\lambda), d(\mu), d(\nu) \leq k$. Then:

$$(6.1) \quad g(\lambda, \mu, \nu) \leq \frac{1}{k^8 k^2 28 k^3} n^{4k^3 + 13k^2 + 31k}.$$

We also know that the actual maximal value is not much further from this bound. Let

$$A(n, k) := \max \{g(\lambda, \mu, \nu) : \lambda, \mu, \nu \vdash n \text{ and } \ell(\lambda), \ell(\mu), \ell(\nu) \leq k\}.$$

Theorem 6.9 ([PP23]). For all $k \geq 1$, there is a constant $C_k > 0$, such that

$$(6.2) \quad A(n, k) \geq C_k n^{k^3 - 3k^2 - 3k + 3} \quad \text{for all } n \geq 1.$$

However, we are nowhere close to finding asymptotics close to these maxima for concrete families of partitions.

Open Problem 6.10. For every fixed k determine a family of partitions $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)} \vdash n$ with $\ell(\lambda^{(n)}), \ell(\mu^{(n)}), \ell(\nu^{(n)}) \leq k$, such that

$$g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) = O(n^{(k-1)^3}).$$

Using the powerful technique of ACSV (Asymptotic Combinatorics in Several Variables), the cases of $k \leq 4$ have been somewhat well understood and explicit asymptotics derived in [MT]. For arbitrary, but fixed, values of k though the ACSV is no longer applicable and we need other ways to compute this.

The same question pertains to the plethysm coefficients.

Open Problem 6.11. *Show that for every fixed k the maximal plethysm coefficient over partitions with length k grows polynomially in n . In particular, we have*

$$\log \max_{\lambda \vdash n^2, \ell(\lambda)=k} a_\lambda(n[n]) = \Omega(\log(n))$$

It was established in [IP17] that plethysms are bounded above by rectangular Kronecker coefficients,

$$a_\lambda(d[n]) \leq g(\lambda, n^d, n^d)$$

for $\ell(\lambda) \leq m^2$, $\lambda_1 \geq d(n-m)$ and d, n large enough so we are in the stable regime, i.e. the values of the plethysm and Kronecker are constant when only d, n and λ_1 grow. Thus, we would not expect that plethysm coefficients would have larger growth than Kroneckers.

6.3. Tight asymptotics. There are very few nontrivial cases where we have certain explicit formulas for Kronecker and plethysm coefficients and would be able to derive tight asymptotics. One such case is for rectangular Kronecker coefficients $g(\lambda, m^\ell, m^\ell)$ when λ has two rows. We have that, see [PP17a], for $\lambda = (m\ell - n, n)$,

$$g(m^\ell, m^\ell, \lambda) = p_n(\ell, m) - p_{n-1}(\ell, m) = a_\lambda(m[\ell]),$$

where $p_n(\ell, m) = \#\{\alpha \vdash n, \alpha_1 \leq m, \ell(\alpha) \leq \ell\}$ is the number of partitions of k whose Young diagram fits inside an $\ell \times m$ rectangle.

Using this partition counting interpretation, and deriving tight asymptotics for $p_n(m, \ell)$ using tilted geometric random variables, we obtain

Theorem 6.12 ([MPP19a]). *Let $A := \frac{\ell}{m}$ $B := \frac{n-1}{m^2}$. Let c, d be solutions of a certain system of integral equations with parameters A, B . We have*

$$a_\lambda(m[\ell]) = g(\lambda, m^\ell, m^\ell) = p_n(\ell, m) - p_{n-1}(\ell, m) \sim \frac{d}{m} p_{n-1}(\ell, m) \sim \frac{d e^{m[cA+2dB-\log(1-e^{-c-d})]}}{2\pi m^3 \sqrt{D}}.$$

Note that some technical conditions for the theorem to hold also include that ℓ, m grow at the same rate, and also that $\frac{n}{m\ell}$ should be away from $\frac{1}{2}$. When $n \sim \frac{1}{2}m\ell$ we can no longer estimate the difference between p_n 's so tightly and we need a different approach for the following

Open Problem 6.13. *Find tight asymptotics when $\lambda = ([m^2/2], [m^2/2])$ for $g(m^m, m^m, \lambda)$.*

The best lower bound in this case comes from characters and is obtained in [PP17a].

Needless to say, any other explicit tight asymptotic results would be of great interest and, using semigroup property, could lead to improved lower bounds in other regimes.

7. REDUCED KRONECKER COEFFICIENTS

Since the Kronecker coefficients quickly turned out to be difficult to understand, hopes turned towards the *stable Kronecker coefficients* defined as

$$\bar{g}(\alpha, \beta, \gamma) := \lim_{n \rightarrow \infty} g(\alpha[n], \beta[n], \gamma[n]), \quad \alpha[n] := (n - |\alpha|, \alpha_1, \alpha_2, \dots), \quad n \geq |\alpha| + \alpha_1.$$

They generalize the Littlewood-Richardson coefficients, following Theorem 3.2 from [Mur56].

$$\bar{g}(\alpha, \beta, \gamma) = c_{\beta\gamma}^\alpha \quad \text{for } |\alpha| = |\beta| + |\gamma|,$$

and were thereby called “extended Littlewood-Richardson coefficients” in [Kir04]. They have thus occupied an intermediate spot between the Littlewood-Richardson and ordinary Kronecker coefficients, and have been a subject of independent study and interest, see [Mur38, Mur56, Bri93, Val99, Kir04, Kly04, BOR11, BDO15, CR15, Man15, SS16, IP17, PP20b, OZ21]. Together with the ordinary Kronecker coefficients, the problem of finding a combinatorial interpretation has also been part of the agenda, see [Sta00].

The ordinary Kronecker coefficients can be expressed as a small alternating sum of reduced Kroneckers and reduced Kroneckers are certain sums of ordinary Kroneckers, see [BOR11, BDO15]. These relationships show that reduced Kroneckers are also $\#P$ -hard to compute, see [PP20b]. However, it could not be immediately deduced that their positivity is NP -hard, which prompted the following question.

Open Problem 7.1. *Given input λ, μ, ν (in unary), is deciding whether $\bar{g}(\lambda, \mu, \nu) > 0$ NP -hard? Determine whether computing $\bar{g}(\lambda, \mu, \nu)$ is $\#P$ -complete?*

As we noted, the second part of that problem would follow if there is a combinatorial interpretation for them, or for the ordinary Kronecker coefficients.

Some special cases of combinatorial interpretations can be derived from the ordinary Kronecker coefficients. Separately, in [CR15] a combinatorial interpretation was given when μ, ν are rectangles and λ is one row, and other similar cases were derived in [BO05]. Methods to compute them, as well as for the ordinary Kronecker coefficients, have been developed in a series of papers, see [BDO15, BOR11, OZ20, OZ21].

Until recently, it was believed they are strictly better behaved than the ordinary Kroneckers, with the following conjecture appearing in [Kir04, Kly04]

Conjecture 7.2 (Kirillov, Klyachko). *The reduced Kronecker coefficients satisfy the saturation property:*

$$\bar{g}(N\alpha, N\beta, N\gamma) > 0 \quad \text{for some } N \geq 1 \quad \implies \quad \bar{g}(\alpha, \beta, \gamma) > 0.$$

Using the recent positivity results, in particular for rectangular Kroneckers from [IP17], this was also disproved.

Theorem 7.3 ([PP20b]). *For all $k \geq 3$, the triple of partitions $(1^{k^2-1}, 1^{k^2-1}, k^{k-1})$ is a counterexample to the Conjecture. For every partition γ s.t. $\gamma_2 \geq 3$, there are infinitely many pairs $(a, b) \in \mathbb{N}^2$ s.t. (a^b, a^b, γ) is a counterexample.*

As shown in [PP20b], asymptotically, and complexity-wise they behave more similarly to the ordinary Kronecker than to the Littlewood-Richardson coefficients.

Remark 7.4. In the course of writing this paper, the first part of Open Problem 7.1 was settled in [IP23], where it was shown that every Kronecker coefficient is equal to a [not much larger] reduced Kronecker coefficient:

$$g(\lambda, \mu, \nu) = \bar{g}(\nu_1^{\ell(\lambda)} + \lambda, \nu_1^{\ell(\mu)} + \mu, (\nu_1^{\ell(\lambda)+\ell(\mu)}, \nu)).$$

Thus the reduced Kronecker coefficients are just as hard as the Kroneckers, and the complexity results follow from § 4. This also shows that the problems are essentially the same, and the reduced Kronecker coefficients are not easier in general.

8. METHODS

While the concrete open problems inspire work in the area, the pertinent task is to create tools for understanding the Kronecker and plethysm coefficients. Of course, an ultimate tool could be the “combinatorial interpretation”, but this itself has become the hardest problem to understand and its solution requires new tools. So far, as very little progress has been made in this direction, it is clear that we need something new.

Kronecker coefficients can be interpreted via S_n as multiplicities. Via Schur-Weyl duality, they are also dimensions of invariant spaces under GL action. The last interpretation is what gives the semigroup property, something that cannot be seen from the S_n perspective. On the other hand, the transpositional invariance $g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu)$ is visible from their S_n definition. There are many more examples showing the variety of results one can get using different methods and the limitations of each one also. Perhaps the best tool would be able to unify all these views.

Symmetric functions have been one of the most useful tools in Algebraic Combinatorics, and is certainly the author's favorite. The Littlewood-Richardson coefficients are structure constants in the ring of symmetric functions and thus have been very successfully studied using Schur functions. But the Kronecker coefficients cannot be interpreted in that way. Motivated by that, a family of symmetric functions was developed in [OZ20, OZ21]. Symmetric functions have not been very useful in establishing positivity results, because the formulas involved are often alternating sums. However, they are useful for finding asymptotics and bounds as done in [PP20a, PP23] for example. Within integrable probability they have been a lot more successful, from Asymptotic Representation Theory, through lozenge tilings, to Random Matrices, see e.g. [BG15, GP15, BGH] and references therein.

Plethysm coefficients can also be studied using symmetric functions, yet again positivity properties cannot usually be derived this way. For example in [DIP19] symmetric functions were used to derive some explicit formulas for $a_\lambda(d[n])$ with λ a three row partition, and later this method generalized in [FI20] to exhibit an alternating formula and put them in GapP.

A technique to show positivity is the construction of explicit highest weight tableau evaluations, see [Ful00], which is easier for plethysms but also applicable for Kroneckers. For example, this technique was used in [I15] to prove Theorem 5.4, generalized in [LS17], and in [BIP19] to show positive plethysm coefficients.

Useful tools in Representation Theory are induction and restriction of modules, passing to the representation theory of other groups. For example, the character bound 5.3 was obtained by restricting to the alternating group $A_n \subset S_n$, see [BB04, PPV16, PP17a]. Similar methods were developed to extract positivity criteria working with representations over finite fields, see e.g. [BBL]. In another direction, diagram algebras can also be used to study Kronecker, see e.g. [BDO15], and plethysm coefficients, see e.g. [COSSZ22, OSSZ].

Another useful approach comes from discrete geometry. This is not at all surprising given that many objects in representation theory correspond to integer points in polytopes, e.g. SSYTs are integer points in Gelfand-Tsetlin polytopes, see e.g. [DM06] for some complexity connections also. Vallejo used such relations to develop methods of discrete tomography and find a lower bound for Kronecker coefficients via pyramids, see [Val97, Val99, IMW17]. Kronecker coefficients are bounded above by 3d contingency tables with given 2d marginals, as seen in [?, PP20a] to obtain upper bounds. Similar expressions tying them to Kostant partition functions were developed in [MRS21] and used in [MT] to find asymptotics.

Finally, Computational Complexity theory provides some tools besides the framework. The framework is to explain the hardness of the problems and what possible answers to expect. The tools come in the form of gadgets and basic hard/complete problems to reduce to like 3SAT, PerfectMatching, SetPartition etc. Embedding such a complete problem in our problem is an art, very similar to finding injections, see e.g. [GJ79] for some classical results. Other tools come from algebraic complexity and GCT, where the complexity of certain polynomials could imply inequalities between the relevant structure constants.

Despite the variety of questions, methods and applications, the Kronecker and plethysm coefficients are still a mysterious black box. It sometimes gives us presents. But we cannot see inside... just yet.

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